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Asymptotic Integration of A Class of Nonlinear Parabolic Equations

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Abstract: The objective of this research is to give the explicit computation for the normal form and the normalization mapping of a class of the nonlinear parabolic equations developed in.

Key words: Nonlinear parabolic equations, normal form, normalization mapping, asymptotic expansions, asymptotic integration

INTRODUCTION

Let us consider the class of nonlinear parabolic equations:

$$\begin{cases} u_t + Au + f(u) = 0 \\ u(0) = u_0 \end{cases} \quad (1.1)$$

where A is an unbounded linear operator of the domain D(A) and f is a polynomial such that:

$$f(u) = \sum_{n\geq 2}^{2p+1} a_n u^n \text{ , } p\geq 1,\ a_{2p+1} > 0 \quad (1.2)$$

In a previous work[1], we constructed an asymptotic expansion and a normal form for the Eq. (1.1). In a non resonant case, this normal form is a linear parabolic equation

$$U_t + AU = 0 \quad (1.3)$$

In a general case, the normalization mapping denoted by W builded in[1] associate to a function u_0 belonging to a natural space of initial data an element $W(u_0)$ of a Frechet Space $\Im_A$ such that for every solution u(t) of (1.1), the function v(t) = W(u(t))satisfies the following equation in $\Im_A$.

$$v_t + Av + F(v) = 0 \quad (1.4)$$

where the mapping F(v) (defined in section 1) is a generally nonlinear operator which involves only terms corresponding to resonances in the spectrum of A.

Moreover (1.4) is equivalent to an infinite system of non homogeneous linear ordinary differential equations, which can be (elementarily) integrated. More precisely, the components of F can be expressed in terms of a sequence of polynomials $\{P_J\}_{j=1}^{\infty}$ and the operator f.

The objective of this research is to show uniqueness of the polynomial P_j and to give a way to compute them by induction, although the algorithm which gives the P_j is some what complicated.

Finally, we obtained that the normal form of (1.4) is canonical and we characterize the expansion in terms of its coordinates.

The application of the method to other nonlinear equation has been made in.

NOTATIONS AND PRELIMINARIES

Let V and H be two separable Hilbert spaces such that

$$V \subset H \text{ with compact injection,} \quad (2.1)$$

$$V \text{ is dense in } H. \quad (2.2)$$

We denote by $\|.\|$ and $|.|$ the corresponding norms.

Consider the unbounded operator A with a rang in H

$$D(A) = \{u \in V, Au \in H\}. \quad (2.3)$$

Supplying D(A) with the graph norm, A is then an isomorphism of D(A) in H, so there exists a sequence

$$\{\lambda_i\}_{i=1}^{\infty}$$

of eigenvalues of A

$$0 < \lambda_1 < \lambda_2 < \ldots, \quad (2.4)$$

each of a finite multiplicity and also a sequence

$$\{\omega_i\}_{i=1}^{\infty}$$

orthonormal basis (in H and V) of the associated eigenvectors.

$$A\omega_1 = \lambda_1\omega_i,\ i = 1, 2,... \qquad (2.5)$$

The orthogonal projection in H of the linear span(ω_1,...,ω_m) will be denoted by P_m and R_j denotes the orthogonal projection in the eigenspace of $\wedge_j$

$$R_i\omega = \sum_{\lambda_i}(\omega,\omega_k)_H\omega_k \qquad (2.6)$$

Then we have

$$R_jR_k = 0 \text{ si } i\neq j,\ R_1\oplus R_2\oplus....... = 1 \qquad (2.7)$$

We shall also consider the Frechet space containing H

$$\Im_A = R_1H\oplus R_2H\oplus......, \qquad (2.8)$$

equipped with the topology of convergence of components, the operator A and the semi-group, e^{-At} generated by A, extend to $\Im_A$.
We denote by

$$0<\wedge_1<\wedge_2< .... <\wedge_j < ...., \qquad (2.9)$$

The sequence of eigenvalues of multiplicity m_k and by $\{S(t)\}_{t\geq 0}$ the nonlinear semigroup defined by

$$S(t)\colon V\to V,\ u_0\to S(t)u_0 \qquad (2.10)$$

Finally, we call resonance in the spectrum of A, the relation:

$$a_1\wedge_1+a_2\wedge_2+...+a_k\wedge_{k+1},\ a_1\in N,\ i = 1, k \qquad (2.11)$$

We shall denote by

$$0<\mu_1 = \wedge<\mu_2<.... \qquad (2.12)$$

The elements of the additive semi-group $\Im$ generated by $\wedge_k$'s and $k_j = \text{Max}\{k,\wedge_k\leq\mu_j\}$.
On the other hand, we define the power A^α of the operator A, for $\alpha\in\Re$ and we denote by

$$V_\alpha = D(A^{\frac{\alpha}{2}}) \text{ where } V_0 = H, V_1 = V \qquad (2.13)$$

Supplying V_α with the following norm

$$|u|_\alpha = \left|A^{\frac{\alpha}{2}}u\right|_0 \qquad (2.14)$$

then V_α is a Hilbert space.
Finally, we introduce a sequence

$$\{E_m\}_{m=1}^{\infty}$$

of Hilbert space such that

$$E_{m+1}\subset E_m\forall m$$
with continuous injection (2.15)

$$\forall m,\ V_m \text{ is close subspace of } E_m \qquad (2.16)$$

The following theorem showed some of the main results of[1].

Theorem 2.1:

i) There exists a one to one analytic mapping

$$W\colon V\to\Im_A$$

satisfying W'(0) = 1, such that for every regular solution u(t) of (1.1), v(t) = W(u(t)) satisfies the equation

$$V_t(t)+Av(t)+F(v(t)) = 0 \qquad (2.17)$$

where

$$F_k(v)=R_kF(v)=\sum_{\mu_{\alpha_1}+....+\mu_{n_0}=\wedge_k} R_k\left[P_{\alpha_1}\left(v_1,...,v_{k_{\alpha_1}}\right)...P_{\alpha n0}\left(v_1,...,v_{k_{\alpha n0}}\right)\right] \qquad (2.18)$$

and

$$V = v_1\oplus v_2\oplus... \in\Im_A \qquad (2.19)$$

ii) In (2.18), for j = 1,2,...,P_j are $E_m\cap D(A)$ value polynomials defined on

$$R_1H \oplus R_2H \oplus ... \oplus R_{k_j}H$$

depending on the spectrum of and. Such that

$$W_{\mu_j}(0,u_0) = P_j\left(W_1(u_0),...,W_{k_j}(u_0)\right)$$

where

$$W_1(u_0) = R_1W_{\wedge_1}(0,u_0)$$

Moreover, if

$$M(X_1,...,X_{k_j})$$

is a monomial in

$$P_j(X_1,...,X_{k_j})$$

of degree

$$m_1,...,m_{k_j}$$

in

$$X_1,...,X_{k_j}$$

respectively, then

$$m_1\Lambda_1 + ... + m_{k_j}\Lambda_{k_j} = \mu_j .$$

Furthermore if μ_j is eigenvalue Λ_k i.e.,

$$\mu_j = \Lambda_{k_j}$$

then

$$P_j(X_1,...,X_k) = X_k + \text{higher order terms in } X_1,...,X_{k-1}$$

We recall that, the mapping W was constructed in[1] from expansion of every regular solution of (1.1), the following theorem gives the properties of the expansion which will be used in the present research.

Theorem 2.2: For each n∈N, the solution u(t) of (1.1) admits the following expansion in H

$$u(t) = W\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t} + ... + \mu_N e^{-\mu_N t} + V_N(t) \quad (2.20)$$

where $W\mu_j(t)$ is a $E_m \cap D(A)$-valued polynomial in t and $V_N(t) \in ([0,\infty),v) \cap C^{\infty}((t_1,\infty),E_m \cap D(A))$, $t_1>0$,. $M \geq 0$.
This expansion satisfies the following properties:

- $|V_N(t)|_m = O\left(e^{-(\mu_N+\varepsilon_N)t}\right)$, $\forall \varepsilon_N > 0$, $m \geq 0$.
- $d_j^0 = \deg W\mu_j(t) \leq j-1$, j = 1,2,...,N.
- if $\Lambda_j \leq \mu_N$, $\Lambda_j \in \sigma(A)$ a non resonant eigenvalue, then $W_{\Lambda j}$ is a constant in t and $R_j W_{\Lambda_j} = W_{\Lambda_j}$.
- if $\mu_j \leq \mu_N$ is not a nonresonant eigenvalue, $W_{\mu j}$ satisfies the equation

$$\frac{d}{dt}W_{\mu_j} + (A-\mu_j)W_{\mu_j}(t) + \sum_{\mu_{\alpha_1}+....+\mu_{n_0}=\mu_j} W_{\mu_{\alpha_1}} \bullet ... \bullet W_{\mu_{\alpha_{n_0}}} \quad (2.21)$$

If Λ_j is a resonant eigenvalue we have

$$\deg W\mu_j(t) \leq 1 + \max_{\mu_{\alpha_1}+....+\mu_{n_0}=\mu_j} \left(d^0\alpha_1 + ... + d^0\alpha_{n_0}\right) \quad (2.22)$$

Moreover,

$$R_k W_{\mu_j(t)}$$

for k≠j and the order of the coefficient will obtained from

$$R_{\Lambda_1}W_{\Lambda_1}(0),...,R_{\Lambda_{j-1}}W_{\Lambda_{j-1}}(0)$$

- For any μj∈ℑ

$$W_{\mu_j}(0,u_0) = P_j\left(W_1(u_0),...,W_{k_j}(u_0)\right) \quad (2.23)$$

where P_j are the polynomial defined in theorem 1.1 and $W_k(u_0) = R_{\Lambda k}W_{\Lambda k}(0,u_0)$ are the components of the normalizing mapping W.

- Let $\prod_n$: $\Im_A \to R_1H \oplus R_2H \oplus ... \oplus R_NH$ be the canonical projection N = 1,2,... then the range of $\prod_n W$ contains a ball centred at 0.

Remark 2.1:

We will henceforth use the notation

$$S(t,u_0) \approx \sum_1^{\infty} W_{\mu_j}(t,u_0) e^{-\mu_j t} \quad (2.24)$$

for the asymptotic expansion (2.19).

THE ALGEBRA OF THE ASYMPTOTIC EXPANSION

Proposition 3.1: For every u∈V, the asymptotic expansion (2.19) can be rewritten as follows

$$S(t)u \approx \sum_{j=1}^{\infty} P_j\left(W_1(S(t)u),...,W_{k_j}(S(t)u)\right) \quad (3.1)$$

Proof: We denote $k_j = \{k, \Lambda_k \leq \mu_j\}$;

Let, according to (2.23), has the asymptotic expansion

$$S(t)u \sim \sum_{j=1}^{\infty} e^{-\mu_j t} W_{\mu_j}(t,u) \quad (3.2)$$

It results from theorem 2.2

$$W_{\mu_j}(0,u) = P_j\left(W_1(u),...,W_{k_j}(u)\right) \quad (3.3)$$

We also have for any t, t≥0

$$S(t+t_0,u) \approx \sum_{j=1}^{\infty} e^{\mu_j t} W_{\mu_j}(t+t_0,u) \quad (3.4)$$

Hence that

$$S(t+t_0,u) \sim \sum_{j=1}^{\infty} e^{\mu_j t} W_{\mu_j}(t,S(t_0)u) \quad (3.5)$$

By the uniqueness of the expansion (2.19), it follows that

$$W_{\mu_j}\left(t,S(t_0)u\right) = e^{-\mu_j t_0} W_{\mu_j}(t+t_0,u) \quad (3.6)$$

Which for t = 0, we obtain

$$W_{\mu_j}\left(0,S(t_0)u\right) = e^{-\mu_j t_0} W_{\mu_j}(t,u) \quad (3.7)$$

So from theorem 2.1, we have

$$W_{\mu_j}\left(t,u\right) = e^{-\mu_j t} P_j\left(W_1\left(S(t)u\right),...,W_{k_j}\left(S(t)u\right)\right) \quad (3.8)$$

We deduce then from (2.23) and (3.8), the formula

$$S(t)u) \sim \sum_{j=1}^{\infty} e^{-\mu_j t} W_{\mu_j}(t,u) = \sum_{j=1}^{\infty} P_j\begin{pmatrix} W_1\left(S(t)u\right) \\ ,...,W_{k_j}\left(S(t)u\right)\end{pmatrix} \quad (3.9)$$

The next proposition gives further properties of the P_j.

Proposition 3.2:

- If $\mu_j \in \Im$ is the eigenvalue Λ_{kj}, then

$$P_j\left(X_1,X_2,...,X_{k_j}\right) = X_{k_j} + Q_{k_j}\left(X_1,X_2,...,X_{k_{j-1}}\right) \quad (3.10)$$

where

$$d^0 Q_{k_j} \geq 2 \text{ and } R_k Q_{k_j}\left(X_1,X_2,...,X_{k_{j-1}}\right) = 0 \quad (3.11)$$

- For $u \in V$, we denote $X_k(t) = W_k(u(t))$, then for every j

$$\sum_{m=2}^{k_j}\left(D_m P_j\right) F_m = \left(A-\mu_j\right)P_j + \sum_{\mu_{\alpha_1}+....+\mu_{n_0}=\mu_j} P_{\alpha 1}...P_{\alpha n_0} \quad (3.12)$$

where D_m is the derivative with respect to the m^{th} component.

Remark 3.1: In (3.12) the polynomials are evalued at

$$X_1(t),...X_{k_j}(t),$$

but the formula is valid for arbitrary

$$X_1,...X_{k_j}$$

since W`(0) = 1.

Proof: Part i) results from the theorem 1.1 and[1]. Part ii) from (2.18) and (3.9), we derive for $\mu_j \in \Im$, the Eq

$$\frac{d}{dt}\left(e^{\mu_j t} P_j\begin{pmatrix} X_1,X_2, \\ ...,X_{k_j}\end{pmatrix}\right) + \left(A-\mu_j\right)e^{\mu_j t} P_j\begin{pmatrix} X_1,X_2, \\ ...,X_{k_j}\end{pmatrix} + \sum_{\mu_{\alpha_1}+....+\mu_{n_0}=\mu_j} e^{\mu_j t} P_{\alpha_1}\left(X_1,...,X_{\alpha_1}\right)...P_{\alpha_{n_0}}\left(X_1,...,X_{\alpha_{n_0}}\right) \quad (3.13)$$

From theorem 2.1, the Eq. (3.13) can be rewritten as

$$\frac{d}{dt} P_j\begin{pmatrix} e^{\Lambda_1 t}X_1,... \\ ,e^{\Lambda_{k_j} t}X_{k_j}\end{pmatrix} + \left(A-\mu_j\right)P_j\begin{pmatrix} e^{\Lambda_1 t}X_1, \\ ...,e^{\Lambda_{k_j} t}X_{k_j}\end{pmatrix} + \sum_{\mu_{\alpha_1}+....+\mu_{n_0}=\mu_j} P_{\alpha_1}\begin{pmatrix} e^{\Lambda_1 t}X_1, \\ ...,e^{\Lambda_{\alpha_1} t}X_{\alpha_1}\end{pmatrix} \cdot ... \cdot P_{\alpha_{n_0}}\begin{pmatrix} e^{\Lambda_1 t}X_1, \\ ...,e^{\Lambda_{k_{n_0}} t}X_{\alpha_{n_0}}\end{pmatrix} \quad (3.14)$$

Differentiation, $e^{\Lambda_k t}X_k(t)$, k = 1,2,...., with respect to t, we get

$$\frac{d}{dt}\left(e^{\Lambda_k t}X_k(t)\right) = \Lambda_k e^{\Lambda_k t}X_k(t) + e^{\Lambda_k t}\dot{X}_k(t) \quad k = 1,2,.... \quad (3.15)$$

and from (2.17) we have

$$\frac{dX_k}{dt} = \dot{X}_k(t) = -\Lambda_k X_k - F_k(X(t)) \quad (3.16)$$

then

$$\frac{d}{dt}\left(e^{\Lambda_k t}X_k(t)\right) = -\dot{X}_k(t) = -e^{\Lambda_k t}F_k \quad (3.17)$$

Coming back to (3.14), we deduce then

$$\sum_{m=1}^{k_j}\left(D_m P_j\right) F_m = \left(A-\mu_j\right)P_j + \sum_{\mu_{\alpha_1}+....+\mu_{n_0}=\mu_j} P_{\alpha 1}...P_{\alpha n_0} \quad (3.18)$$

where the summation starts at m = 2 since $F_1 = 0$.

CONSTRUCTION OF THE P_J

We denote by

$$R_{\mu_j} = R_k$$

if $k = k_j$ and

$$R_{\mu_j} = 0$$

otherwise.

In the other hand we set

$$Q_{\mu_j} = I - R_{\mu_j}$$

and setting

$$\overset{0}{X}_1 \in R_1 H,\ \overset{0}{X}_2 \in R_2 H,\ \ldots$$

The following theorem gives the main result concerning the construction of the P_j.

Theorem 4.1: We define a sequence of polynomials q_j as follows

$$q_1(t) = \overset{0}{X}_1 \tag{4.1}$$

and by induction

$$\begin{cases} q_j(t) = -\sum_{n\geq 0}(1-)^n\left[\left(A-\mu_j\right)\right]^{-n-1}\left(\frac{d}{dt}\right)^n \\ \sum_{\mu_{\alpha_1}+\ldots+\mu_{n0}=\mu_j} q_{\alpha_1}\ldots q_{\alpha_{n0}} \\ \qquad \text{if } \mu_j \notin \sigma(A) \end{cases} \tag{4.2}$$

or,

$$\begin{cases} q_j(t) = \overset{0}{X}_{k_j} + \int_0^t \sum_{\mu_{\alpha_1}+\ldots+\mu_{n0}=\mu_j} q_{\alpha_1}\ldots q_{\alpha_{n0}} R_{k_j}\begin{bmatrix} q_{\alpha_1}(\tau)+ \\ \ldots + q_{\alpha_{n0}}(\tau)\end{bmatrix} \\ +\sum_{n\geq 0}(-1)^n\left(A-\mu_j\right)\left(I-R_{k_j}\right)^{-n-1}\left(\frac{d}{dt}\right)^n \times \\ \sum_{\mu_{\alpha_1}+\ldots+\mu_{n0}=\mu_j}\left(I-R_{k_j}\right)\left(q_{\alpha_1}\ldots q_{\alpha_{n0}}\right) \quad \text{if } \mu_j = \mu_{k_j} \end{cases} \tag{4.3}$$

then

$$\begin{aligned} q_j(t) = P_{j0}\left(\overset{0}{X}_1,\ldots,\overset{0}{X}_{k_j}\right) + tP_{j,1}\left(\overset{0}{X}_1,\ldots,\overset{0}{X}_{k_j}\right) \\ + t^{j-1}P_{j,j-1}\left(\overset{0}{X}_1,\ldots,\overset{0}{X}_{k_j}\right) \end{aligned} \tag{4.4}$$

where the $P_{j,i}$ are polynomials maps in

$$\overset{0}{X}_1,\ldots,\overset{0}{X}_{k_j}$$

and

$$P_j\left(\overset{0}{X}_1,\ldots,\overset{0}{X}_{k_j}\right) \equiv P_{j,0}\left(\overset{0}{X}_1,\ldots,\overset{0}{X}_{k_j}\right)$$

Proof: To prove theorem 4.1, we have need the following algebraic lemma.

Lemma 4.1: Let L an inversible (not necessarily bounded) linear operator in some Hilbert space H. Then for any H-valued polynomial r(t). The equation

$$q` = Lq + r \tag{4.5}$$

has a unique polynomial solution given by

$$q(t) = \sum_{k\geq 0}(-1)^k L^{-k} r^{(k)}(t) \tag{4.6}$$

Proof of lemma 4.1: We deduce after an immediate computation that (4.6) is a solution of the Eq. (4.5).

To proof the uniqueness of q, we consider q_1 and q_2 two polynomial solutions of (4.5), the difference $q = q_1 - q_2$ is solution of equation

$$q` = Lq \tag{4.7}$$

Since q is a polynomial, then $q^{(n)} = 0$ for $n > d^0 q$ which with (4.7) yields $0 = Lq^{(n-1)}$ and since L is inversible, we obtain $q^{(n-1)} = 0$.

Inductively, we have $q = 0$, so $q_1 = q_2$ and we have the uniqueness of q.

Let us take back the proof of theorem 4.1 and one assumes that $\mu_j \notin \sigma(A)$.

Let us consider the equation when q_j is a polynomial of t.

$$\begin{aligned} q_j'(t) + \left(A-\mu_j\right)q_j(t) + \sum_{\mu_{\alpha_1}+\ldots+\mu_{n0}=\mu_j} q_{\alpha_1}(t)\ldots q_{\alpha_{n0}}(t) = 0 \\ q_j(0) = P_j\left(\overset{0}{X}_1,\ldots,\overset{0}{X}_{k_j}\right) \end{aligned} \tag{4.8}$$

We remark that, the Eq. (4.8) is exactly of type (4.5) where $-L = A-\mu_j$ and the second member

$$b = \sum_{\mu_{\alpha_1}+\dots+\mu_{n_0}=\mu_j} q_{\alpha_1}\dots q_{\alpha_{n_0}}.$$

It follows from the lemma 4.1 that

$$q_j(t) = -\sum_{n\geq 0}(1-)^n\left[\left(A-\mu_j\right)\right]^{n-1}\frac{d^n}{dt^n}\sum_{\mu_{\alpha_1}+\dots+\mu_{n_0}=\mu_j} q_{\alpha_1}\dots q_{\alpha_{n_0}} \quad (4.9)$$

and of course

$$\begin{aligned} q_j(t) &= P_{j0}\left(\overset{0}{X}_1,\dots,\overset{0}{X}_{k_j}\right)+tP_{j,1}\left(\overset{0}{X}_1,\dots,\overset{0}{X}_{k_j}\right) \\ &+t^{j-1}P_{j,j-1}\left(\overset{0}{X}_1,\dots,\overset{0}{X}_{k_j}\right) \end{aligned} \quad (4.10)$$

To prove (4.3), we assume that

$$\mu_j = \Lambda_{k_j} \in \sigma(A)$$

and we set

$$q_j(t) = \beta_j(t) + Q_{\Lambda k_j} q_j(t) \quad (4.11)$$

where

$$\beta_j(t) = R_{\Lambda k_j} q_j(t) \in R_{\Lambda k_j} H$$

and

$$Q_{\Lambda k_j} = \left(I - R_{\Lambda_{k_j}}\right)\cdot$$

Coming back to (4.8) and using (4.11), we obtain

$$\beta_j'(t) + \sum_{\mu_{\alpha_1}+\dots+\mu_{n_0}=\mu_j} R_{k_j}\left[q_{\alpha_1}\cdot\ldots\cdot q_{\alpha_{n_0}}\right] = 0 \quad (4.12)$$

and

$$\begin{aligned} &Q_{\Lambda k_j} q_j(t) + \left(A-\mu_j\right)\left(Q_{\Lambda k_j} q_j\right) \\ &+\sum Q_{\Lambda k_j}\left[q_{\alpha_1}\cdot\ldots\cdot q_{\alpha_{n_0}}\right] = 0 \end{aligned} \quad (4.13)$$

The eq. (4.12) will be integrated

$$\beta_j(t) = \beta_j(0) + \int_0^t \sum_{\mu_{\alpha_1}+\dots+\mu_{n_0}=\mu_j} R_{\Lambda_{k_j}}\left[q_{\alpha_1}(\tau)\dots q_{\alpha_{n_0}}(\tau)\right]d\tau \quad (4.14)$$

and it follows that

$$R_{\Lambda_{k_j}} q_j(t) = \overset{0}{X}_{k_j} + \int_0^t \sum_{\mu_{\alpha_1}+\dots+\mu_{n_0}=\mu_j} R_{\Lambda_{k_j}}\left[q_{\alpha_1}(\tau)\dots q_{\alpha_{n_0}}(\tau)\right]d\tau \quad (4.15)$$

The Eq. (4.13) is exactly of type (4.5), or its results from lemma 4.1 that

$$\begin{aligned} &\left(I-R_{\Lambda_{k_j}}\right)q_j = -\sum_{n\geq 0}(1-)^n\left[\frac{\left(A-\mu_j\right)}{\left(I-R_{\Lambda_{k_j}}\right)}\right]^{-n-1}\left(\frac{d}{dt}\right)^n \times \\ &\sum_{\mu_{\alpha_1}+\dots+\mu_{n_0}=\mu_j}\left(I-R_{\Lambda_{k_j}}\right)\left(q_{\alpha_1}\dots q_{\alpha_{n_0}}\right) \end{aligned} \quad (4.16)$$

Finally, we deduce (4.2) and (4.3) from (4.15) and (4.16).
Hence the theorem 4.1 is competing.

Corollary 4.1: The polynomials P_j are unique.

CARACTERIZATION OF THE EXPANSION IN TERMS OF ITS COORDINATES

We recall that, a previous work, we constructed a set of analytic nonlinear manifolds.

$$M_k = \{u \in V, W_1(u) = \dots = W_{k-1}(u) = 0\} \quad (5.1)$$

that are invariant under S(t), of codimension $m_1+\dots m_k$ and we have the condition

$$S(t) = O\left(e^{-\Lambda_{k+1}t}\right) \quad (5.2)$$

Also setting $M_0 = V$, u satisfies

$$u \in M_{k-1}\setminus M_k \Leftrightarrow \begin{cases} |S(t)u| = O\left(e^{-\Lambda_{k+1}t}\right) \\ \text{and} \\ e^{-\Lambda_{k+1}t} = O\left(S(t)u\right) \end{cases} \quad (5.3)$$

Consequently, if $u \in M_{k-1}$, then the expansion (3.1) takes the form

$$S(T)u \sim \sum_{k_j\geq k} P_j(0,\dots,0,W_k(S(t)u),\dots,W_{k_j}(S(t)u)) \quad (5.4)$$

However, for N fixed and $K_N \geq k$, let d_N denoted the degree of the polynomial

$$\sum_{\substack{k_j \geq k \\ j \leq N}} P_j(0,...,W_k(v),...,W_{k_j}(v)) \qquad (5.5)$$

then it follows from ii) of theorem 2.1 that

$$d_N \leq \frac{\mu_N}{\Lambda_k}$$

so if $v \in M_{k-1} \backslash M_k$, then we can write

$$\left| v - \sum_{\substack{k_j \geq k \\ j \leq N}} P_j(0,...,W_k(v),...,W_{k_j}(v)) \right| = O\left(|v|^{d_N + \frac{\varepsilon_N}{\Lambda_k}} \right) \qquad (5.6)$$

where ε_N is as theorem 2.2 i). In particular if $v \in M_0 \backslash M_1$ then

$$\left| v - \sum_{j=1}^{N} P_j(W_k(v),...,W_{k_j}(v)) \right| = O\left(|v|^{d_N + \frac{\varepsilon_N}{\Lambda_1}} \right) \qquad (5.7)$$

where d_N is the degree of the polynomial

$$\sum_{j=1}^{N} P_j \quad \text{in} \quad W_1(v),...,W_{k_N}(v)$$

Hence, the expansion (5.4) can be viewed as an asymptotic expansion of v = S(t)u in terms of its coordinates $W_k(v)$, $W_{k+1}(v)$,.... .

Finally, notice that the asymptotic expansion (5.6), (5.7) hold the trajectories but we don't know it they are global or not.

Remark: Present research extends to the equation with second member such that

$$u_t + Au + f(u) = g(x)$$

where g(x) is independent of time, for example:

$$\begin{cases} \dfrac{d\omega}{dt} - \Delta\omega + 3u_\infty^2\omega + 3u_\infty\omega^2 + \omega^3 = 0 & \Omega \times]0,T[\\ \omega(0) = \omega_0 \quad \Omega; \quad \omega = 0 & \partial\Omega \times]0,T[\end{cases}$$

where $\omega = u - u_\infty$ with u and u_∞ are, respectively solution of equations

$$\begin{cases} u_t - \Delta u + u^3 = g(x) & \Omega \times]0,T[\\ u(0) = u_0 \quad \Omega; \quad u = 0 & \partial\Omega \times]0,T[\end{cases}$$

and

$$\begin{cases} -\Delta u_\infty + u_\infty^3 = g & \Omega \\ u_\infty = 0 & \partial\Omega \end{cases}$$

We obtain the problem (1.1) with:

$$\begin{cases} Au = -\Delta u + 3u_\infty^2 u \\ f(u) = 3u_\infty u^2 + u^3 \end{cases}$$

REFERENCES

1. Aissaoui, M.Z., 1987. Comportement asymptotique et forme normale pour une classe d'équations paraboliques abstraites. thèse de Doctorat, Université de Paris IX, Centre d'Orsay.