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On the Minimum Energy of a Linear Discrete Neutral System with Delayed Control

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Abstract: Criteria for existence and the form of an optimal control are presented. It is shown that if the discrete system is relatively null controllable with constraints, the optimal control that drives the system to the origin of E^n is unique and Bang-Bang.

Key words: Minimum energy, linear, discrete neutral system, delay control

INTRODUCTION

Optimal control generally means controlling a system in a "best way". It ensures that a system gets to its target with a minimum consumption of energy and at minimum time. Sparingly on the literature are the works of Lasalle (1959),Chuckwu (1982) and Onwuatu. In his study on the time optimal control problem of linear neutral functional systems, Chukwu (1982) talked the system

$$\frac{d}{dt}D(t,x_t) = L(t,x_t) + B(t)u(t) \qquad t \geq 0 \qquad (1.1)$$

Where the control set is a unit n-dimensional cube and the target is a continuous set function in an n-dimensional Euclidean space. He provided necessary and sufficient conditions for the existence and uniqueness of optimal controls.

In a follow-up work, Onwuatu treated the optimal control of discrete systems with delays using the system

$$x(t) = Ax(t) + \sum_{j=1}^{p} B_j x(t-j) + \sum_{j=1}^{p} D_j u(t-j), \qquad t\varepsilon[0,t_1]$$
$$x(t)=\phi(t), \quad t\varepsilon[-p,0] \qquad (1.2)$$

He established sufficient conditions for relative null controllability and also necessary conditions for an optimal control.

This research is aimed at looking into a system, which could be regarded as putting systems (1.1) and (1.2) together in perspective to obtain a linear discrete neutral system with delay in the control thus complementing both results.

PRELIMINARIES, DEFINITIONS AND NOTATIONS

We shall consider the following autonomous discrete neutral system with delayed control defined by

$$\frac{d}{dt}[x(t) - A_{-1}x(t-h)] = A_0 x(t) + \sum_{i=1}^{N} A_i x(t) + \sum_{i=1}^{N} B_i u(t-i) \qquad x(t) = \phi(t),\ t\varepsilon\ [-N,0] \qquad (2.1)$$

Where x(t) is a measurable m-vector continuous function. A_{-1}, A_0, A_i are nxn matrices. B_i are nxm constant matrices and $\phi(t)$ is a continuous vector function in [-N,0]. The control function u(t) ε E^n is assumed to be measurable and bounded on every finite interval. Here $E=(-\infty,\infty)$, the real line and E^n the n-dimensional Euclidean space with norm $\|\cdot\|$. We let

C = C([-N,0], E^n) be a Banach space of continuous functions and we designate the norm of am element in C by

$$\|\phi\| = \sup_{-N\leq S\leq 0} |\phi(s)|.$$

We let $L_1([a,b],E^n)$ be the space of Lebesgue integrable functions taking [a,b] into E^n with

$$|\phi\| = \int_a^b |\phi(s)|ds,\ \phi\ \varepsilon\ L_1([a,b],E^n)$$

If x ε $C([a,b],E^n)$, for any a≤ b, then for each fixed t ε [a,b], the symbol x_t denotes an element of C given by

$$x_t(s) = x(t+s), \quad -N\leq s\leq 0$$

The function u(t) is similarly defined throughout the sequel . The controls of interest are

- $B = L_\infty([0,t_1],E^m)$
- $U = L_\infty([0,t_1],C^m)$

Where $C^m = \{u{:}u\ \varepsilon\ E^n, |u_j|\leq 1, j=1,\ldots, m\}$

$L_\infty([a,b],E^m)$ is the space of essentially bounded functions taking [a,b] to En with the norm

$$\|\phi\| = \operatorname{ess\,Sup}_{S\varepsilon[a,b]} |\phi(s)|.$$

The above conditions on A_{-1}, A_i and B_i ,i=1,…,N, ensure that for each initial data (0,ϕ), a unique solution of (2.1) exists through (0,ϕ), (Chukwu, 1982) which is continuous on (0,ϕ). This solution is given by

$$x(k,o,\phi,u) = X(k,0)\phi(0) + \sum_{i=1}^{N}\int_{-i}^{0} X(k,t+i)A_i x(t)dt + \sum_{i=0}^{N}\int_{-i}^{0} X(k,t+i)B_i u_9(t)dt + \sum_{i=0}^{N} X(k,t+i)B_i u(t)dt \quad (2.2)$$

Where X(t,s) is the fundamental matrix solution of the free system of (2.1) which satisfies the equation

$$\frac{\partial}{\partial t}X(t,s) - A_{-1}\frac{\partial}{\partial t}X(t-h,s) = \sum_{i=1}^{N} A_i x(t) \quad t > s \quad (2.3)$$

$$\text{where} \quad X(t,s) = \begin{cases} 0 & s-h \le t \le s \\ 1 \ldots\ldots\ldots\ldots & t = s \end{cases}$$

We now define the controllability gramian W(0,k) as follows

$$W(k,0) = \int_0^k \left[\sum_{i=0}^{N} X(k,t+i)B_i\right]\left[\sum_{i=0}^{N} X(k,t+i)B_i\right]^T dt \quad (2.4)$$

Where T denotes matrix transposition. We also define the reachable set by

$$|R = \left\{\int_0^k \sum_{i=0}^{N} X(k,t+i)Bu(t)dt; \; u \in |B\right\} \quad (2.5a)$$

In particular, we define the constrained reachable set as

$$|R^+ = \left\{\int_0^k \sum_{i=0}^{N} X(k,t+i)Bu(t)dt; \; u \in |U\right\} \quad (2.5b)$$

The following properties of the reachable set must be noted

- 0 ε |R(k,0) for each k>0
- The reachable set is symmetric, compact and convex for k≥0
- X(k,s)|R(s,0) ⊆ |R(k,0), 0≤ s≤ k

Proof: The Bang-Bang principle also holds for system (2.1) and is stated as follows:
Let

$|U^0$={u/u measurable, $|u_j(t)|=1$, j=1,….,m, tε$[0,t_1]$ }

(These are controls which at all times utilize all the control available.)

Then,

$$|R^0 = \left\{\sum_{i=0}^{N}\int_0^{k-i} X(k,t+i)B_i u^0(t)dt\right\} \; u^0 \, \varepsilon |U^0$$

$|R^0(k,0)$ are the set of points reachable by the Bang-Bang control.

Proposition 1: (The Bang-Bang Principle):

$$|R(k,0) = |R^0(k,0) \text{ for all } k \ge 0$$

Proof: Because X(k,t+i) ε $L_\infty([0,t],E^{nxm})$ and B_i ε $L_1([0,t],E^{nxm})$, we have $X(k,t+i)B_i$ ε $L_1([0,t],E^{n2})$, It follows from LaSalle (1959) that |R(k,0) = $|R^0(k,0)$ for each k.

Definition 2.1: (Complete state): The complete state at time t for system (2.1) is defined as

y(t) = {x(t),u(t)}

where $u_t(\theta) = u(t+\theta)$, θ ε [-N,0]

Definition 2.2:(Relative controllability): System (2.1) is said to be relatively controllable on $[0,t_1]$ if for every complete state y(0) = $\{x(0),\Phi,u_0\}$ and every $x_1 \varepsilon E^n$, there exists a control u ε |B such that the corresponding trajectory of system (2.1) satisfies $x(t_1,0,\Phi,u) = x_1$.

Definition 2.3: (Relative null controllability): System (2.1) is said to be relatively null controllable at t = t_1 if for any initial state y(0) = $(x_0,\Phi.u_0)$, there exists an admissible control u ε |B defined on $[0,t_1]$ such that the response of system (2.1) satisfies $x(t_1,0,\Phi,u) = 0$.

System (2.1) is said to be relatively null controllable at t = t_1 with constraints if rather u ε|U.

EXISTENCE OF OPTIMAL CONTROL

Let z(t) ε E^n be a target point function which is time-varying, we now prove a theorem for the necessary conditions for the existence of an optimal control.

Theorem 3.1: Assume system (2.1) is relatively controllable to the target, then there exists an optimal control.

Proof: The variation of constant formula for system (2.1) is given by

$$x(k,0,\phi,u) = X(k,0)\phi(0) + \sum_{i=0}^{N}\int_{-i}^{0} X(k,t+i)A_i x(t)dt + \sum_{i=0}^{N} X(k,t+i)B_i u(t)dt + \sum_{i=0}^{N} X(k,t+i)B_i w(t)dt \quad (3.1)$$

Controllability to the target is equivalent to x (t_0,0, Φ,u) = z(t_1) for some t_1.That is

$$w(t_1) = x(t_1) - X(t_1,0)\phi(0) - \sum_{i=1}^{N}\int_{-i}^{0} X(k,t+i)A_i x(t)dt - \sum_{i=9}^{N}\int_{-i}^{0} X(t,t-i)B_i u_0(t)dt + \sum_{i=0}^{N}\int_{0}^{t_1-i} X(t,t+i)B_i u(t)dt \quad (3.2)$$

This is equivalent to w(t_1) ε |R(t_1,0). Let t* = inf{t:w(t) ε |R(t_1,0). Clearly, we have 0≤ t* < t_1.

There is a non-increasing sequence of times t_n converging to t* and a sequence of controls u^n ε L_∞([0,t_1],E^n).With

$$w(t_n) = y(t_n,u^n) = \sum_{i=0}^{N}\int_{0}^{t_n-i} X(t_n.t+i)B_i u^n(t)dt \in |R(t_n,0)$$

But

$$\| w(t^*) - y(t^*,u^n)\| \le \| w(t^*) - u(t_n)\| + \| u(t_n) - y(t^*,u^n)\| \le \| w(t^*) - w(tn) | + J$$

Where
J = || w(t_n) - y(t*,u^n)||

$$\le \left\| \sum_{i=0}^{N}\int_{0}^{t_n-i} X(t_n,t+i)B_i u^n(t)dt - \sum_{i=0}^{N}\int_{0}^{t^*-i} X(t_n,t+i)B_i u^n(t)dt \right\| + \left\| \sum_{i=0}^{N}\int_{0}^{t^*-i} X(t_n,t+i)B_i u^n(t)dt - \sum_{i=0}^{N}\int_{0}^{t^*-i} X(t^*,t+i)B_i u^n(t)dt \right\|$$

$$= \left\| \sum_{i=0}^{N}\left[\int_{0}^{t_n-i} X(t_n,t+i)B_i u^n(t)dt - \int_{0}^{t^8-i} X(t_n,t+i)B_i u^n(t)dt \right] \right\| + \left\| \sum_{i=0}^{N}\left[\int_{0}^{t^*-i} X(t_n,t+i)B_i u^n(t)dt - \int_{0}^{t^*-i} X(t^*,t+i)B_i u^n(t)dt \right] \right\|$$

$$= \sum_{i=0}^{N}\left\| \left[\int_{0}^{t_n-i} X(t_n,t+i)B_i u^n(t)dt - \int_{0}^{t^*-i} X(t_n,t+i)B_i u^n(t)dt \right] \right\| + \sum_{i=0}^{N}\left\| \left[\int_{0}^{t^*-i} X(t_n,t+i)B_i u^n(t)dt - \int_{0}^{t^*-i} X(t^*,t+i)B_i u^n(t)dt \right] \right\|$$

By Cauchy-Schwartz inequality, the last equation above is less than or equal to

$$\sum_{i=0}^{N}\left\| \int_{t^*-i}^{t_n-i} X(t_n,t+i)B_i u^n(t)dt \right\| + \sum_{i=9}^{N}\left\| \int_{0}^{t^*-i} \left[X(t_n,t+i) - X(t^*,t+i) \right] B_i u^n(t)dt \right\| \quad (3.4)$$

But X(t_n,t+i)$B_i u^n$ is integrable and [t_n-i,t*-i] < ∞. And so the first term on the right hand side of the inequality tends to zero as t_n→t*. Also, X(t_n,t+i)→X(t*,t+i) in the uniform topology of E^n.. Hence by the boundedness convergence theorem, the second summand on the right hand side tends to zero as n → ∞.

Since solutions are continuous as well as the target,||u(t*) - u(t_n)|| →0 as t_n→ t*.

Hence, w(t*) = $\lim_{n\to\infty} y(t^*,u^n)$. But |R(t,0) is closed by proposition 1 and y(t*,u^n) ε |R(t*,0),

w(t*) = y(t*,u*) for some u*ε |U and by definition of t*, u* is optimal.

NECESSARY CONDITIONS FOR OPTIMAL CONTROL

We now return to our original goal of hitting a continuously moving target z(t) in minimum time.

Consider the trajectory of system (2.1) given by

$$x(k,o,\phi,u) = x(k,0)\phi(0) + \sum_{i=1}^{N}\int_{-i}^{0} X(k,t+i)A_i\phi(t)dt + \sum_{i=1}^{N}\int_{-i}^{0} X(t,t+i)B_i u_0(t)dt + \sum_{i=0}^{N}\int_{-i}^{0} X(t,t+i)B_i u(t)dt \quad (4.1a)$$

or equivalently

$$w(k) = z(k) - X(k,0)\phi(0) - \sum_{i=1}^{N}\int_{-i}^{0} X(k,t+i)A_i\phi(t)dt - \sum_{i=1}^{N}\int_{-i}^{0} X(k,t+i)B_i u_0(t)dt - \sum_{i=0}^{N}\int_{0}^{k-i} X(k,t+i)B_i u(t)dt \quad (4.1b)$$

Then reaching z(t) in time k corresponds to

$$z(k) - X(k,0)\phi(0) - \sum_{i=1}^{N}\int_{-i}^{0} X(k,t+i)A_i\phi(t)dt - \sum_{i=0}^{N}\int_{-i}^{0} X(k.t+i)B_i u_0(t)dt \equiv w(k)\ \varepsilon R(k,0)$$

We show that if u* is the optimal control with t* the optimal time, then

$$z(t^*) - X(t^*,0)\phi(0) - \sum_{i=1}^{N}\int_{-i}^{0} X(t^*,t+i)A_i\phi(t)dt - \sum_{i=0}^{N}\int_{-i}^{0} X(t^*,t+i)B_i u_0 dt \equiv u(t^*)\ \varepsilon\partial|R(t^*,0)$$

that is u(t*) is on the boundary of the constrained reachable set.

Theorem 4.1: Let u* be the optimal controlwith t* the minimum time. Then u(t*)ε ∂|R(t*,0)m the boundary of |R(t*,0)

Proof: Assume u* is used to hit w(t) in time t*. Then

$$z(t^*) - X(t^*,0)\phi(0) - \sum_{i=1}^{N}\int_{-i}^{0} X(t^*,t+i)A_i\phi(t)dt$$
$$-\sum_{i=0}^{N}\int_{-i}^{0} X(t^*,t+i)B_i u_0(t)dt \equiv u(t^*)\varepsilon|R(t^*,0)$$

Assume u(t*) is not on the boundary, then u(t*) ε Int |R(t*,0), t* > 0. Hence, there exists a ball B(u(t*),r) of radius r about u(t*) such that B(u(t*), r)⊆|R(t*,0). Because |R(t,0) is a continuous function of t, there exists δ > 0 such that B(u(t*), r) ⊆ |R(t,0) for t* -δ≤ |t| ≤ t*. Therefore, u(t*) ε |R(t,0), t* - δ≤ t. This contradicts the optimality of t*. Hence, u(t*) ⊆∂ |R(t*,0).

Theorem 4.2: If u* be an optimal control transferring system (2.1) from y(0) to z(t*) in minimum time t*, then there exists a non zero function η ε E^n such that

$u^*(t) = \text{sgn}\{\eta^T X(T,t)B\} = \text{sgn}\{\eta^T X(T-t)B$

where X(T,t) is the solution of the adjoint equation

$$y(t) = -y(t)A - \sum_{i=1}^{N} y(t,i)B \qquad t\varepsilon[0,T]$$
$$y(T) = y0, \qquad y(t) = 0 \quad t > 1$$

and

$$B = (b_1, b_2, ..., b_N) = \sum_{j=0}^{N} X(t,t+j)B_j$$

Proof:

Define y(t) = X(T,t)b_i and

$$u(t^*) = z(t^*) - X(t^*,0)\varphi(0) - \sum_{j=0}^{N}\int_{-i}^{0} X(t^*,t+1)A_j\phi(t)dt$$
$$-\sum_{j=1}^{N}\int_{-i}^{0} X(t^*,t+1)B_j u_0(t)dt \tag{4.3}$$

That is

$$u(t^*) = \sum_{j=0}^{N}\int_{0}^{t^*-j} X(t^*,t+j)B_j u^*(t)dt$$
$$= \sum_{j=9}^{N}\int_{0}^{t^*-j} X(t^*,t)X(t,t+i)B_j u^*(t)dt \tag{4.4}$$

Set $\quad B = \sum_{j=9}^{N} X(t,t+j)B_j$

then $\quad u(t^*) = \int_{0}^{t^*-j} X(t^*,t)Bu^*(t)dt$

From the hypotheses and theorem 4.1, u(t*) is on the boundary ∂|R(t*,0) of the constrained reachable set. The supporting hyperplane theorem (Hermes and Lassale, 1969) then implies the existence of a non trivial hyperplane with outward normal η (say) supporting |R(t*,0) at u (t*). In other words $\eta^T u(t^*) \geq \eta^T y$ for all yε|R(t*,0) and y≠0; that is

$$\eta^T \int_{0}^{t^*-j} X(t^*,t)Bu^*(t)dt \geq \eta^T$$
$$\int_{0}^{t^*-j} X(t^*,t)Bu(t)dt \text{ for all } u\,\varepsilon\,U$$

Rearranging gives

$$\eta^T \int_{0}^{t^*-j} X(t^*,t)B[u^*(t) - u(t)]dt \geq 0$$

This can happen only if

$$u^* = \text{sgn}\{\eta^T X(t^*,t)B\} \tag{4.5}$$

CONCLUSION

From this, we see that controllability results with controls constrained to lie in a compact subset |U of E^n are very useful in the resolution of time optimal control problems.

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