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Optimal Control of Neutral Systems with Nonlinear Base (The Maximum Principle Perspective)

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Abstract: In this research we examine the Euclidean controllability for a neutral system with a nonlinear base given by

$$* \frac{d}{dt} D(t,x_t) = f(t,x_t, u(t)) + B(t)u(t)$$

By a careful analysis of the maximum principle, necessary and sufficient conditions for the existence and uniqueness of optimal controls are deduced. This research is a great improvement of existing works providing a relationship between the attainable and reachable sets.

Key words: Optimal control, netural system, nonlinear base, perspective

INTRODUCTION

Optimal controls in its simplest sense means controlling a system in some "best way", Little wonder; interest is intense in this area. The purpose of this research is not unconnected with the growing interest in the realm (Banks and Jacobs, 1973; Chukwu, 1982; Hale and Grux, 1971; Mlirza and Womack, 1972; Onwuatu, 2000). For linear systems of the form:

$$\frac{d}{dt}D(t_1x_t) = L(t,x_t) + B(t)u(t), \quad t > o \tag{1}$$

Chukwu (1982) gave necessary and sufficient conditions for the existence and uniqueness of optimal control. More reports for system (1) are available in Banks and Kent (1972) and Banks and Jacobs (1973).

In (1978) Galh studied the Euclidean controllability of nonlinear perturbations of linear functional differential systems of neutral type. In his investigation, the base is inherently linear and controllable and the perturbations are assumed to satisfy some growth conditions. In contrast to this research, our current study assumes that f and g may be nonlinear in the case of time delay systems with $g(t,\phi) = 0$. Mlirza and Womack (1972) studied conditions under which the system is Euclidean null controllable,

$$\frac{d}{dt}D(t, x_t) = L(t,x_t) + B(t)u(t) \tag{2}$$

that is x = 0 in definition (5). The present endeavor is to investigate Euclidean controllability of the system (*) and set ahead to obtain Optimal conditions for the system with straightforward application to the Maximum principles. Then we shall establish a relationship between the attainable and reachable sets.

NOTATIONS AND DEFINITIONS

We consider system (*) given by

$$\frac{d}{dt}D(t_1x_t) = f(t,x_t,u(t)) + B(t)u(t) \qquad t \in I$$
$$x = \phi \in C([-h,o],E^n) \equiv C_1$$

Where the operator D is given by

$$D(t, \phi) = \phi(o) - g(t, \phi) \tag{3}$$

and

$$g : [\delta,t_1] \text{ x } C_1 \to E^n, \qquad t_1 \in I$$
$$f : [\delta,t_1] \text{ x } C_1 \text{ x } C([d,t_1],E^n) \to E^n$$

are continuous functions. B is a continuous n×m matrix function

Here E= (-∞,∞), E^n is a real n- dimensional linear vector space with norm $|\cdot|$ $C([a,b],E^r)$ is the Banach

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space of continuous functions mapping [a,b} into E^n with the suprimum norm denoted for ϕ ([a,b], E^r) by $\|\phi\|=\sup(a\leq s\leq b)\|\phi(s)\|$ D is as defined in (Daurr, 1978) If

$$\delta\in E, \tilde{c}, \geq 0 \text{ and } x\in C([\delta-h,\delta+c],E^n),$$

Then for any $t\in[\delta,\delta+\tilde{c}]$, the function $x_t\in C$ is defined by

$$x_t(s) = x(t+s), -h\leq s\leq 0 \quad (4)$$

Only continuous function $u\in C([\delta,t_r],E^n)$ are considered. We shall assume that f(t, ϕ, u) is uniformly Lipschitzian with respect to ϕ and u for any $t\in I$ with Lipschitz constant k. In the sequel $C_1 = C_1$ ([-h,o}, E^n) and C_2 ([δ,t_1], E^n).

Thus f: $I\times C_1\times C_2\rightarrow E^n$ and g: $I\times C_1\rightarrow C^n$ Also, if U is an open subset of I xC_1xC_2, then v is the corresponding projection onto $I\times C_1$. That is v = {(t, ϕ); (t, ϕ, u) $\in$ U}.

Definition: Let v be an open subset on I x C_1 Let

$$\bar{\phi}(t,\phi,u\, s)=\{\psi\in C_1{}':(t,\psi)\in v,$$

$\|\Psi-\phi\|\leq\mu$. $\Psi(\theta)\Rightarrow(\theta)\ \theta_1-s,\ \theta\in[-h,0)\}$ then g:v-> E^n is monatomic at zero if for any (t_1 ϕ)$\in$v, there exists $s_0 = s(t,\phi)>0$, $\mu_0=\mu(t,\phi)>0$,continuous in t, ϕ and a scalar function $\rho(t,\phi,u,s)$ defined and continuous for (t,ϕ)$\in$v, $0\leq s\leq s_0$, $o\leq\mu\leq\mu_0$ nondecreasing in μ, s such that.

$$\rho(t,\phi,\mu_0,s_0)<1,\ |g(t,\Psi)-g(t_1\phi)\leq\rho(t_1\phi,u,s)\|\Psi-\phi\| \\ \text{for } t\in E, \psi\in\bar{\phi}(t_1,\phi,u,s) \text{ and all } 0\leq s\leq s_0 \text{ and } 0\leq\mu\in\mu_0 \quad (5)$$

Definition: Given $\delta\in E$, $\phi\in C$, we say x(δ,ϕ) is a solution of (*) with initial value ϕ and δ if there exists

$$a\,\tilde{c}>0 \text{ such that } x\in C([\delta-h,)(\delta+\tilde{c}],E^n)$$

x coincides with ϕ on [(δ-h,δ] and D (t, x_t) is continuously differentiable on $[(\delta,\delta+\tilde{c}]$ and satisfies system (*) on $\{\delta,\delta+\tilde{c}]$. It is known (Hale and Grux, 1971) that under the prevailing assumptions on D, *f*, g, B and u for each $\phi\in C_1$, there is a unique solution of system (0.1) with initial value ϕ at δ.

Definition (Euclidean controllability): The system (*) is Euclidean controllable if for any initial function $\phi\in C_1$ and any vector $x_1\in E^n$ there exist some $t_1<\tilde{c}$ and a control $u\in C$ ([δ,t_1]E^n) such that the solution x(t) = (t, δ,ϕ,u) of (*) exists and satisfies $x(t_1) = x_1$.

Lemma 1: Let B: $I\rightarrow E^{nxm}$ be a continuous function. Assume that f: $I\times C_1\times C_2\rightarrow E^n$ is uniformly lipschtzian, that is,

$$|f(t,\phi_1,u_1)-f(t,\phi_2,u_2)|\leq k\left[\|\phi_1-\phi_2\|+\|u_1-u_2|\right]$$

for all $t\in I$, $\phi_1,\phi_2\in C_1, u_1,u_2,\in C_2$
Then the constants t_1, v can be chosen such that:

- $(1-v)(1+\|B^*(t)B^{-1}(\delta,t_1-\delta)\|)<½$
- $\|B^*(t)B^{-1}(\delta,t_1-\delta)\|t_1\ k<½$ for all $t\in I$

Definition (Attainable set): The attainable set of system (*) at time t. denoted by A(t) is defined as the set of all those points $u\in E^n$ for which the system can be steered in time t by the use of all admissible controls u. that is

$$A(t) = \{x(t,\delta,\phi,u), u\in U\}$$

Definition (Reachable set): By setting

$$\Re(t)+\int_{t_0}^{t} y(s)ds\, u\in U$$

Where $y(t) = (F^{-1}(t)B(t)$, we call $\Re(t)$ the reachable set of system (*) in time t. Clearly $\Re(t)\in E^n$. Both the attainable and reachable sets are related and they jointly contribute in the establishment of optimality for the control system.

Theorem 1: (The maximum principle)

RESULTS AND DISCUSSION

Theorem: A function x is a solution of system (*) through (δ,ϕ) if and only it there exists a $\tilde{c}>o$ such that x satisfies the equation

$$D(t,x_t)=D(\delta,\phi)+\int_\delta^t f(s,xs,u(s)ds+ \\ \int_\delta^t B(s)u(s)ds, t\in[o,\tilde{c}]\ x_\delta=\phi) \quad (6)$$

Proof: Since D $(t,x_t) = x(t)-g(t,x_t)$ we deduce that, the solution of (*) is given by

$$x(t)=D\delta,\phi)+g(t,x_t)+\int_\delta^t f(s,x_s,u(s)ds+ \\ \int_\delta^t B(s)u(s)ds, t\geq\delta) \quad (7)$$

We observe from (7) that x is a solution of (*) on [δ,δ+t] if and only if

$$x(\delta+t)=\bar{\phi}(t)+z(t), -h\le t\le \tilde{c} \quad (8)$$

Where z(t)satisfies.

$$z(t)=g(t+\delta,\phi_t+z_t-g(\delta_1\phi)+\int_0^t f(s+\delta,\phi_s+z_s,u(s+\delta))ds + \int_0^t B(s+\delta)u(s+\delta)ds, z_0=0 \quad (9)$$

We note that for $t_1\in(\delta_1,\tilde{c}), x(t_1)=x_t$ if an only if

$$z(t_1-\delta) = x_1 - \phi(t_1-\delta) \quad (10)$$

The corresponding u which steers ϕ to x_1 in time t_1 is given by

$$u(t)=B^*(t)H^{-1}[x_1-\bar{\phi}(t_1-\delta)+g(\delta,\phi) - g(t_p-\bar{\phi}_{t1}-\delta+z_{t1}-\delta]-\int_\delta^{1-\delta} f(s+\delta,\bar{\phi}_s+\bar{z}_s,u(s+\delta))ds \quad (11)$$

Where

$$H=H(\delta,t_1-\delta)=\int_\delta^{1-\delta} B(s+\delta)B^*(s+\delta)ds$$

Such a u can be shown to exist under the conditions imposed on f,g and B by the method of Banks and Jacobs (1973), Banks and Kent (1972). Moreover this function is unique and is defined for all $t\in[\delta,\tilde{c}]$

Now set

$$\begin{cases} a)\ T(z,u)(t)=o \text{ if } t\in[-h,o] \\ b)\ S(z,u)(t)=o \text{ if } t\in[-h,o] \\ c)\ T(z,u)(t)=(Y(t),W(t)), \text{if } t\in[o,\tilde{c}) \\ d)\ S(z,u)(t)=(h(t),o), \text{if } t\in[o,\tilde{c})\equiv I \end{cases} \quad (12)$$

Where

$$Y(t)=g(t+\delta,\bar{\phi}_t z_t)-g(\delta,\phi)$$

$$h(t)=\int_0^t f(s+\delta,\bar{\phi}_s+z_s,u(s+\delta)ds+\int_0^t B(s+\delta)u(s+\delta)ds)$$

$$W(t)=B^*(t)H^{-1}[x_1-\phi(t_1-\delta)+g(\delta,\phi) - g(t_1,\phi_{t1},\delta+z_{t1}-\delta)-\int_0^{t_1-\delta} f(s+\delta,\bar{\phi}_s+z_s,u(s+\delta))ds]$$

From the above remarks it is clear that if the operator T+S given by (T+S) (z,u) (t) = (Y(t) + h(t),w(t)),has a fixed point so that (T+S) (z,u) (t) =(z(t),u(t)) then the system (*) is Euclidean controllable. In that case z(t) is given by (9) and u(t) by (11)

Theorem 3: If u^* is an optimal control which steer the system (*) from a point x_0 to the point x_1 in the state space E^n and if t^* is the minimum time to achieve this, then there exists a non zero vector $K\in E^n$ such that

$$u^*(t) = \text{sgn}\, [k^T F^{-1}(t)\, B(t)] \quad (13)$$

Proof: We wish to minimize t_1 such that $x_1\in A$ (t) Now suppose we set

$$y(t) = F^{-1}(t)\, B(t) \quad (14)$$
$$\text{or } y(t) = (y_1(t), y_2(t) \ldots\ldots\ldots y_n(t))^T$$

Where T denotes matrix transposition and suppose for t^* the optional time,

$$w_1 = F^{-1}(t^*)x_1 - x_0 \in E^n$$

$$\text{then} \quad w_1 = \int_{t_0}^{t_1^*} y(s)u^*(s)ds$$

From definition (5), it easy to see that, in particular $\Re(o) = \{\bar{O}\}$ and $\Re(t)$ increases with time and meets w_1 at the boundary of R (t^*). Now at w_1.there exists a hyper plane of support for $\Re(t)$ with outward normal K at w_1. Thus for minimum time and for $K\neq 0$, we have $K^TW_1>K^TW$ for all $w\in\Re(t)$ This implies

$$\int_{t_0}^{t^*} K^T y(s)[u^*(s)-u(s)]\, ds \ge o \quad (15)$$

Now from a consideration of $u^* = \pm 1$ it is easy to see that the above inequality is satisfied if u^* and $K^Ty(s)$ have the same sign simultaneously. Hence, we infer that $u^* = \text{sgn}\,(k^Ty(s))$.

Which from (14) is true for all $t\in t\ [0,t^*]$. That is Eq. (13) holds. To harness the maximum principle for this purpose, we define a Hamiltonian function $\bar{H}$ given by

$$\bar{H} = (x,u\ \phi) = \phi\, f(x,t,u) + \phi_0\, f \text{or } \phi_0 \ge 0 \text{ and} \quad (16)$$

$\bar{H}^*(t_1x,u) = \text{Max}\ \bar{H}\ (t,x,u)$. A necessary condition that u^* be optimal and $\phi^*(t_1 u)$ the corresponding optimal path is that there exists a non zero vector function ϕ such that

$$\bar{H}(t,u,\phi) = \bar{H}^*(t, u^*, \phi^*) \text{ and } \dot{\phi} = -\frac{\partial H}{\partial x},\ \dot{x} = \frac{\partial H}{\partial \phi_s}$$

$$\bar{H} = \phi_0 + \phi\, Bu(t)$$

$$\bar{H}^* = \max_u \{\phi_0 + \phi\, Bu(t)\}$$

$$= \phi_0 + \max_u \{\phi\, Bu\,(t)\}$$

Since the solution $\phi(t)$ is given by $\phi(t) = \{K^T F^{-1}(t)$ for $K\in E^n$ we deduce that H^* can have its maximum if u* = sgn

$\{K_T\ x^{-1}\ B(t)\}$ as in (13). However it is assumed that f satisfies all smoothness conditions for existence and uniqueness of solutions and that $f(t,0,0)=0$

Remark 1: A special relationship exists between the attainable set A(t) and the reachable set $\Re(t)$ as in definitions (4) and (5) as follows:

$$A(t) = x(t,u) = F(t)x_o + \int_{t_o}^{t_1} F^{-1}(t-s)B(s)ds$$
$$= F(t) + \int_{t_o}^{t_1} F_1(s)B(s)u(s)ds$$
$$= F(t)x_o + \int_o^t y(s)u(s)ds$$

$$A(t) = F(t)\ x_o + \Re(t) \quad (18)$$

Remark 2: (The bang bang principle): With reachable set $\Re(t,0) \subseteq E^n$, exploit could be made of the bang bang principle whose immediacy of applicability is not guaranteed were $\Re(t,0)$ is subset of a function space. Define the Bang-Bang controls on [0,t]; $t_1 > 0$ by $C^m = \{u{:}u$ is measurable, $/u_j(t)/=1$ J=1,2,3, in $t \in [0,t_1]$

$$\Re^o(t,0) = \{\int_{t_o}^{t} F(0,s)B(s)u(s)\ ds;\ u \in C^m\}$$

The principle states; $\Re(t,0) = \Re^0(t,0)$

Proof: Chukwu (1982).

CONCLUSION

Optimal control literally means controlling a system in a "best way". This has been observed in much control linear processes of certain types. Exploits are now directed to nonlinear systems where unavoidable nonlinearities in systems affect the evolution of the system in a direct manner. This work x-rays and resolves such nonlinearities by considering it as a base for a linear system and thereafter fixes it to zero. A special case of the maximum principle is proved for system (*). Necessary conditions and form of the optimal control deduced. More interestingly the neutral control system is shown to be not only Euclidean controllable also but optimally controllable.

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