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Uniqueness of Solution of Upper Convected Maxwell Flows Through Some Uniform Tubes

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Abstract: The unsteady flow of fluids in tubes of uniform circular cross-sections has applications in the chemical and petroleum industries. A significant amount of work, mainly theoretical, has been done for Newtonian and non-Newtonian flows. In this study, a particular non-Newtonian fluid is examined, which falls into the class of viscoelastic fluids and is known as the Upper Convected Maxwell fluid. In 1990, Han Shifang and Wo Yueqing, examined the transient response of a flow of this viscoelastic fluid in a tube of uniform circular cross-section. In more recent times, for this class of viscoelastic fluids, Rahaman and Ramkissoon examined some basic but interesting pipe flows. In neither research did the researchers investigate the fundamental question of the uniqueness of the solution. This research establishes that the solution for such is in fact unique.

Key words: Uniqueness, upper convected maxwell, viscoelastic, uniform tubes, circular, rectangular

INTRODUCTION

Unsteady pipe flows are of importance to scientists and engineers mainly because of their widespread applications. A tremendous amount of work, mainly theoretical, has been done both for Newtonian and non-Newtonian flows (Balmer and Fiorina, 1980; Etter and Schowalter, 1965; Fan and Chao, 1965; Gorla, 1981; Gorla and Madden, 1984; Ting, 1964).

Shifang and Yueqing (1990) examined the transient response of a flow of an Upper Convected Maxwell fluid in a tube of uniform circular cross-section. Rahaman and Ramkissoon (1995) examined the flow of this same fluid, subjected to various pressure gradients, through a pipe of uniform circular cross-section.

Rahaman (1997) examined the flow of this Upper Convected Maxwell fluid through a duct with a uniform rectangular cross-section, here the flow due to particular time dependent pressure gradients were obtained and quantities such as the drag and friction factor computed. In none of these studies was the fundamental question of the uniqueness of solution investigated, which is now investigated.

MATERIALS AND METHODS

The rheological equation of state for the Upper Convected Maxwell fluid Eq. 7 is given by:

$$\underset{\sim}{T} = -p\underset{\sim}{I} + \underset{\sim}{S} \tag{1}$$

$$\underset{\sim}{S} + \lambda \breve{\underset{\sim}{S}} = 2\mu \underset{\sim}{D} \tag{2}$$

Where:

- $\underset{\sim}{T}$ = Total stress
- $\underset{\sim}{S}$ = Extra stress tensor
- $\underset{\sim}{D}$ = Deformation rate tensor
- p = Isotropic pressure
- λ = Relaxation time
- μ = Viscosity coefficient
- $\vee$ = Represents the upper-convected derivative, defined by:

$$\breve{\underset{\sim}{S}} = \breve{S}^{ij} = \frac{\partial S^{ij}}{\partial t} + q^m S^{ij}{}_m - S^{im} q^j{}_m - S^{mj} q^i{}_m \tag{3}$$

The dynamic equation is:

$$\nabla \bullet \underset{\sim}{S} - \nabla p = \rho \frac{d\underset{\sim}{q}}{dt} \tag{4}$$

Where:

- $\underset{\sim}{q}$ = Velocity field

In the case of unsteady, incompressible axially symmetric flows in tubes of uniform circular and rectangular cross-sections, the constitutive and dynamic Eq. 1, 2 and 4 lead to the field Eq. 8:

$$\frac{\partial w}{\partial t} + \lambda \frac{\partial^2 w}{\partial t^2} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\lambda}{\rho}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right) + \upsilon \nabla^2 w \tag{5}$$

where, w is the component of the velocity in the axial z-direction; while the continuity equation takes the form:

$$\nabla . \underset{\sim}{q} = \frac{\partial w}{\partial z} = 0 \quad (6)$$

Equation 5 and 6 must be solved subject to relevant initial and boundary conditions. However, the fundamental question that should first be answered is the question of uniqueness of solution of this system, which is now addressed.

RESULTS AND DISCUSSION

Uniqueness theorem: Consider a more general system of equations:

$$\frac{\partial q_i}{\partial t} + \lambda \frac{\partial^2 q_i}{\partial t^2} = -\frac{1}{\rho} p_{,i} - \frac{\lambda}{\rho} \frac{\partial}{\partial \underset{\sim}{t}}(p_i) + \upsilon q_{i,jj} \quad (7)$$

$$q_{i,i} = 0 \quad (8)$$

and assume it represents an incompressible, unsteady viscoelastic flow in a bounded region V with fixed boundary S and that the field parameters ($\underset{\sim}{q}$, p), satisfy the following conditions:

- q_i, $\partial q_i/\partial t$ and p are known throughout the closure of V at t = 0
- $\partial q_i/\partial t$ bounded at all times
- q_i is prescribed on S at all times
- p is continuous in V and its first order spatial derivatives are bounded in the closure of V at all times
- q_i and their first and second derivatives are bounded continuous functions in V

Theorem: The system of equations given by Eq. 7 and 8, subject to the conditions (i) to (v) has a unique solution.

Proof: Assume there are two possible solutions $(\underset{\sim}{q}_\alpha, \underset{\sim}{p}_\alpha)$ and $(\underset{\sim}{q}_\beta, p_\beta)$ and let

$$\underset{\sim}{Q} = \underset{\sim}{q}_\beta - \underset{\sim}{q}_\alpha, \; P = p_\beta - p_\alpha$$

We shall proceed to show that $\underset{\sim}{Q} = 0$ and P = 0 in V. Substituting $(\underset{\sim}{q}_\alpha, \underset{\sim}{p}_\alpha)$ and (q_β, p_β) into Eq. 7 and subtracting gives,

$$\frac{\partial Q_i}{\partial t} + \lambda \frac{\partial^2 Q_i}{\partial t^2} = \frac{1}{\rho} P_i - \frac{\lambda}{\rho} \frac{\partial}{\partial t}\left(P_i\right) + \upsilon Q_{i,jj} \quad (9)$$

Multiplying Eq. 9 by Q_i and integrating over V gives,

$$\int_V \frac{\partial Q_i}{\partial t} Q_i dV + \lambda \int_V \frac{\partial^2 Q_i}{\partial t^2} Q_i dV = -\frac{1}{\rho} \int_V P_i Q_i dV - \frac{\lambda}{\rho} \int_V \frac{\partial}{\partial t}(P_i) Q_i dV + \upsilon \int_V Q_{i,jj} Q_i dV \quad (10)$$

Similarly, Eq. 8 gives,

$$Q_{i,i} = 0 \quad (11)$$

Observe that,

$$\int_V \frac{\partial Q_i}{\partial t} Q_i dV = \frac{1}{2} \frac{\partial}{\partial t} \int_V Q_i Q_i dV \quad (12)$$

$$\int_V Q_i \left(\frac{\partial^2 Q_i}{\partial t^2} \right) dV = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int_V Q_i Q_i dV - \int_V \frac{\partial Q_i}{\partial t} \frac{\partial Q_i}{\partial t} dV \quad (13)$$

$$\int_V \frac{\partial}{\partial t}(P_i) Q_i dV = \int_V \frac{\partial}{\partial t}(P_i Q_i) dV - \int_V P_i \frac{\partial Q_i}{\partial t} dV \quad (14)$$

Using Eq. 12 in 10 gives,

$$\frac{1}{2} \frac{\partial}{\partial t} \int_V Q_i Q_i dV \le \lambda \left| \int_V \left(\frac{\partial^2 Q_i}{\partial t^2} \right) Q_i dV \right| + \frac{1}{\rho} \left| \int_V P_{,i} Q_i dV \right| + \frac{\lambda}{\rho} \left| \int_V \frac{\partial}{\partial t}(P_{,i}) Q_i dV \right| + \upsilon \left| \int_V Q_{i,jj} Q_i dV \right| \quad (15)$$

Equation 13 gives,

$$\left| \int_V \left(\frac{\partial^2 Q_i}{\partial t^2} \right) Q_i dV \right| \le \frac{1}{2} \frac{\partial^2}{\partial t^2} \int_V Q_i Q_i dV + \frac{N_1}{\lambda} \int_V Q_i Q_i \, dV \quad (16)$$

where the boundedness condition (v) has been used so that

$$\left| \int_V \frac{\partial Q_i}{\partial t} \frac{\partial Q_i}{\partial t} dV \right| \le \frac{N_1}{\lambda} \int_V Q_i Q_i \, dV$$

in which N_1 is an appropriate large number. Using again the boundedness conditions on P and q_I gives:

$$\frac{1}{\rho} \left| \int_V P_{,i} Q_i dV \right| \le N_2 \int_V Q_i Q_i \, dV \quad (17)$$

Using Eq. 14 gives,

$$\frac{\lambda}{\rho}\left|\int_V \frac{\partial}{\partial t}(P_i)Q_i dV\right| \le \frac{\lambda}{\rho}\left|\int_V \frac{\partial}{\partial t}(P_i, Q_i)dV\right| + \frac{\lambda}{\rho}\left|\int_V P_i \frac{\partial Q_i}{\partial t} dV\right|$$

Using the boundedness conditions here gives,

$$\frac{\lambda}{\rho}\left|\int_V \frac{\partial}{\partial t}(P_i)Q_i dV\right| \le N_2 \frac{\partial}{\partial t}\int_V Q_i Q_i dV + N_3 \qquad (18)$$
$$\int_V \left|\frac{\partial Q_i}{\partial t}\right| dV \le N_4 \frac{\partial}{\partial t}\int_V Q_i Q_i dV$$

where, N_2 , N_3 and N_4 are appropriate large numbers. Similarly,

$$\upsilon\left|\int_V Q_{i,jj} Q_i dV\right| \le N_5 \int_V Q_i Q_i dV \qquad (19)$$

where again, N_5 is a relatively large number.

Substituting Eq. 16-19 into 15 and letting

$$I = \int_V Q_i Q_i dv, I' = \frac{\partial I}{\partial t}, I'' = \frac{\partial^2 I}{\partial t^2}$$

gives

$$-\frac{\lambda}{2} I'' + AI' + BI \le 0 \qquad (20)$$

where A and B and constant

Taking the Laplace transform of (20) and using the conditions,

$$I(0) = I'(0) = 0$$

gives

$$\left[-\frac{\lambda}{2}s^2 + AS + B\right]\bar{I} \le 0 \qquad (21)$$

where, $\bar{I}$ represents the Laplace transform of I. Since, Eq. 21 is valid for all s and fixed λ, A and B ⇒ $\bar{I} = 0$.

CONCLUSION

By uniqueness of Laplace transforms, I = 0⇒ since V is arbitrary, $Q_i \equiv 0$ throughout V for all t. Using this in Eq. 9 and the condition that P = 0 initially, gives P ≡ 0 in V for all t. This establishes the uniqueness property and hence proves the theorem.

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