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Group Representations and Complexity Theory: A Review

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Abstract: Establishing lower bounds for standard algebraic operations is one of the most challenging tasks in Theoretical Computer Science. In this study, researchers discuss lower bounds for matrix multiplication. After a historical review of the progress in this problem. Researchers review a recent development highlighting how the techniques of Group Representation Theory have been applied to give a new approach. Infact, this approach has led to two conjectures whose resolution would achieve the ideal bound for matrix multiplication.

Key words: Theoretical Computer Science, algebraic operations, conjectures, resolution, progress, Group Representation Theory

INTRODUCTION

The task of multiplying matrices is one of the most fundamental problems in algorithmic linear algebra. It is classical that brute force matrix multiplication has complexity $O(n^3)$ involving n^2 multiplications and (n-1) additions. Strassen (1969) made the first break through by divide and conquer strategy to achieve a bound of $O(n^{2.81})$. In 1979, Bini introduced the notion of border rank and obtained $\omega<2.78$, Schonhage generalized this notion in 1981 and showed that $\omega<2.548$. In the same study combining his research with ideas by Pan he also showed $\omega<2.522$ later improved by Romani to $\omega<2.517$ (Williams, 2011).

The first result to break 2.5 was by Coppersmith and Winograd who obtained $\omega<2.496$. Strassen (1969) introduced his laser method which allowed for an entirely new attack on the matrix multiplication problem. He also decreased the bound to $\omega<2.479$. About 3 years later, Coppersmith and Winograd combined Strassen's technique with a novel form of analysis based on large sets avoiding arithmetic progressions and obtained the famous bound of $\omega<2.376$ which remained unchanged for decades. Williams (2011) has marginally improved Coppersmith and Winograd bound to 2.3727 by using the 8th tensor power of the Coppersmith and Winograd algorithm (Williams, 2011).

The next significant development is the novel approach by Cohn and Usmans (2003) and Cohn *et al.* (2005) which uses Group Representation Theory. While this gives a new technique for the problem so far it has not replaced Coppersmith and Winograd bound. But this approach still opens up the possibility of proving $\omega = 2$ subject to the resolution of two conjectures proposed in (Cohn *et al.*, 2005) one combinatorial and the other algebraic.

Problem statement: To apply results and techniques of group representation to estimate the complexity of matrix multiplication. This involves finding appropriate families of groups satisfying structural constraints. Experiments are done with GAP (Rainbolt and Gallian, 2003) (Groups Algorithms and Programming) a software tool to explore the improvements obtained on the lower bounds. Formally, let ω be the smallest exponent for which there is an algorithm for multiplying two $n\times n$ matrices using n^ω arithmetic operations. As the product has n^2 entries to be read clearly $\omega\geq 2$.

MATERIALS AND METHODS

A novel approach to the problem of estimating ω was found by Cohn *et al.* (2005) their idea was to use results of Group Representations Theory to perform block matrix multiplication in the group algebra. This achieves reduction in dimension of the matrices in a recursive fashion-recursion achieved in terms of the dimensions of the irreducible representations of groups.

Matrix multiplication via representation theory: To multiply two $n\times n$ matrices A, B the method embeds both A and B in the representation space of a suitable group G. The embedding is done in such a way that both A and B get represented as series of block diagonal matrices of smaller sizes these sizes are actually the dimensions of irreducible representations of G. Thus, the complexity of matrix multiplication is reduced as by making clever choices of G as the dimension of the block matrices are smaller.

The crucial property of the appropriate group to perform the above is called the Triple Intersection Property (TIP) (Rainbolt and Gallian, 2003). Researchers fix a group G and subsets S, T, U in G, $|S| = |T| = |U| = n$.

Researchers index the rows of A by elements of S and the columns of A by elements in T. Researchers index the rows in B by elements in T and the columns of Bby the elements of U. The rows of resulting product matrix C will be indexed by elements in S and the columns by elements of U. Researchers embed A in the group algebra as the element $A = \sum A_{s,t} s^{-1} t$ where s varies over S and t varies over T and B as the element $B = \sum B_{t,u} t^{-1} u^{-1}$ where t varies over T and u varies over U.

Definition: Researchers say the sets S, T, U satisfy the triple intersection productproperty if for all s_1, s_2 in S, t_1, t_2 in T, u_1, u_2 in U, researchers have:

$$s_1^{-1}t_1t_2^{-1}u_1 = s_2^{-1}u_2t_2^{-1}u_1 \text{iff}, \quad s_1 = s_2, t_1 = t_2, u_1 = u_2$$

Theorem: Let S, T, U be three subsets of size n satisfying the triple intersection product property. Embedding A, B as described before C_{su} is the coefficient of $s^{-1}u$ in the product of A B. The above theorem describes a convenient method to multiply two matrices. Find a group G and subsets having the triple intersection property embed the matrices A, B as described. Elements of the group algebra of G, A and B look like block diagonal matrices. It is easier to multiply block diagonal matrices. Now this resulting matrix has a unique expression as a linear combination of the block diagonal matrices coming from terms of the form $s^{-1}u$, so the product C can be written down using O (n^2) moreoperations.

Now researchers explain how Group Representation Theory enters the picture if G has k irreducible representations V_i of size d_i multiplication in the group algebra (Serre, 1977) reduces to multiplication of k matrices of size $d_i \times d_i$, $1 \leq i \leq k$. If the exponent for matrix multiplication is ω, these k matrices can be multiplied using $\sum O(d_i^{\omega})$ arithmetic operations thus we expect to do the matrix multiplication in atmost $O(n^{\omega})$ operations by definition. It is natural to expect $n^{\omega} \leq \sum d_i^{\omega}$. Infact, Cohn and Umans (2003) and Cohn *et al.* (2005) showed.

Theorem: Let S, T, U be three subsets of size n of a group G_k which satisfy the triple intersection product. Let the size of G be n^{α} for some constant α and let d_i denote the dimension of the irreducible representation V_i of G. Then:

$$n^{\omega} \leq \Sigma d_i \omega$$

Example: The family of groups G_k where G_k is the wreath product of $(Z_k)^6$ with Z_2 satisfies TIP (Triple Intersection Property). With appropriately chosen subsets S, T and U. Using the theorem one can show that $\omega \leq 2.91$.

General case: Cohn *et al.* (2005) have actually established a bound $\omega \leq 2.41$ using a deeper study of group representations. Furthermore, they have proposed two conjectures either of which would realise the ideal bound ω = 2. Researchers will discuss one of the conjectures in the following.

RESULTS AND DISCUSSION

Uniquely solvable puzzles: A Uniquely Solvable Puzzle (USP) of width k is a subset $U \subseteq \{1, 2, 3\}^k$ satisfying the following property. For all permutations $\pi_1\ \pi_2\ \pi_3 \in$ Sym (U), eithe r$\pi_1 = \pi_2 = \pi_3$ or else there exist $u \in U$ and $i \in (k)$ such that at least two of $(\pi_1(u))_i = 1$, $(\pi_2(u))_i = 2$ and $(\pi_3(u))_i = 3$ hold (Cohn and Usmans, 2003; Chon *et al.*, 2005). A strong USP is a USP in which the defining property isstrengthened as follows, for all permutations $\pi_1, \pi_2, \pi_3 \in$ Sym(U), either $\pi_1 = \pi_2 = \pi_3$ or else there exist $u \in U$ and $i \in (k)$ such that exactly two of $(\pi_1(u))_i = 1$, $(\pi_2(u))_i = 2$ and $(\pi_3(u))_i = 3$ hold. For example, the following is astrong USP of size 8 and width 6:

3	3	3	3	3	3
1	3	3	2	3	3
3	1	3	3	2	3
1	1	3	2	2	3
3	3	1	3	3	2
1	3	1	2	3	2
3	1	1	3	2	2
1	1	1	2	2	2

Researchers define the strong USP capacity to be the largest constant C such that there exist strong USPs of size $(C\text{-}o(1))^k$ and width k for infinitely many values of k. With this notation the following conjecture implies ω = 2.

Conjecture

The strong USP capacity equals $3/2^{2/3}$: Researchers use USPs to show ω<2.48 (Cohn and Usmans, 2003; Chon *et al.*, 2005).

IMPLEMENTATION

To study groups computationally GAP software (Rainbolt and Gallian, 2003) was used. This provides explicit construction of S, T, U as in triple intersection property, characters of irreducible representations and hence the block sizes of matrices to be multiplied. A sample code is given as: Experiments were conducted for the family of groups $(Z_3)^6$ wreath product Z_2:

S6: = Group ((1, 2), (1, 2, 3, 4, 5, 6))
Sub: = Subgroup (s6, ((1, 4) (2, 5) (3, 6)))

Z_2: = Group ((1, 2))
hom: = Group homomorphism by images (Z_2, sub, Generators of group (Z_2), (1, 4) (2, 5) (3, 6)))
Z_3: = Group ((1, 2, 3))
w: = Wreath product (Z_3, Z_2, hom)

CONCLUSION

Using a more sophisticated construction in (Cohn *et al*., 2005), the researchers show that their technique can be used so that $\omega \leq 2.41$. They also state conjectures which will prove that $\omega \leq 2$.

In future, Group Representation Theory is expected to become an indispensable tool in the toolkit of algorithm designers and complexity theorists (Kumar, 2010).

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