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Constrained Data Interpolation using C^2 Rational Cubic Spline

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Abstract: Constrained data interpolation is important in many sciences and engineering based disciplines. For instance, the robot path problem always can be considered as constrained data interpolation. Thus, this study the constrained data interpolation using C^2 rational cubic spline with three parameters. The data dependent sufficient condition is derived on one parameter meanwhile the other two parameters can be used to modify the interpolating curve. The unknown first derivatives are calculated by solving tri-diagonal systems of linear equations through Thomas's algorithm. From all numerical results, it can be concluded that the proposed scheme research very well and on par with some established schemes.

Key words: Interpolation, schemes, linear, established, robot, scheme

INTRODUCTION

Shape preserving interpolation and approximation are important in many sciences and engineering problems. There are many criteria for shape preserving. For instance, the user maybe requires that the interpolating or the approximating curves or surfaces to preserves the positivity of the data sets. Rainfall distribution is always having positive value and any negativity is not meaningful in term of statistically. Besides that constrained data modeling also very important in robot path problems as well as other engineering applications such as gear and road designs (Simon and Isik, 1991). The most common scheme that can be used for shape preserving interpolation is rational cubic spline with C^1 or C^2 continuity. Abbas *et al*. (2012) studied the constrained surfaces using C^1 rational cubic spline (cubic/cubic) with three parameters and Awang *et al*. (2013) studied the constrained data interpolation by using C^2 rational cubic spline with three parameters (cubic/quadratic). Furthermore, Hussain and Hussain (2006) utilized the C^1 rational cubic spline of Tian *et al*. (2005) (cubic/quadratic) for constrained data interpolation. The main drawback in the research of Hussain and Hussain (2006) is that there are no free parameters for shape modification. Their scheme is not interactive and not suitable for shape preserving interpolation. Shaikh *et al*. (2011) and Sarfraz *et al*. (2015) constructed rational cubic spline for constrained data interpolation both for curves and surfaces. But it suffers from the fact that there is no free parameter (s) to modify the final shape of the interpolating curve and surface. Goodman *et al*. (1991) initiated the idea on the construction of parametrically defined of rational cubic spline with G^1 for constrained data modeling. By Karim and Kong (2014) the C^1 rational cubic spline with three parameters has been used for shape preserving positivity and constrained data interpolation. The main object in this study is the extension of C^2 rational cubic spline developed by Karim *et al*. (2016) for constrained data interpolation. We consider the functional or scalar data set.

The proposed scheme does not involve any knots insertion. Unlike the scheme of Butt and Brodlie (1993) that require one or two knots need to be inserted in order to preserves the shape of the data.

The proposed scheme research equally for both evenly or unevenly spaced data, in contrast Duan *et al*. (2005) scheme only research if the given data are evenly spaced. The proposed scheme has two free parameters while no free parameter in the research of Duan *et al*. (2005). The proposed scheme is able to produce the constrained data interpolation with C^2 continuity.

MATERIALS AND METHODS

Construction of C^2 rational cubic spline: This study is devoted to the construction of C^2 rational cubic spline with three parameters (Karim *et al*. 2016). Given data point and its first derivative, i.e., $\{(x_i, f_i), i = 0, 1, \ldots, n\}$ and $x_0<x_1<, \ldots, <x_n$. For I = 0,1, ..., n-1, let $h_i = x_{i+1}-x_i$, $\Delta_i = (f_{i+1}-f_i)/h_i$ and $\theta = (x-x_i)/h_i$ with $0\leq\theta\leq 1$. On each sub-intervals $x \in [x_i-x_{i+1}]$, i = 0, 1, 2, ..., n-1, the rational cubic spline interpolant with three parameters can be defined as Eq. 1:

$$s(x) \equiv s_i(x) = \frac{A_{i0}(1-\theta)^3 + A_{i1}\theta(1-\theta)^2 + A_{i2}\theta^2(1-\theta) + A_{i3}\theta^3}{(1-\theta)^2\alpha_i + \theta(1-\theta)(2\alpha_i\beta_i + \gamma_i) + \theta^2\beta_i} \quad (1)$$

The C^2 continuity can be stated as Eq. 2 (Karim *et al*. 2016):

$$\left.\begin{aligned} &s(x_i) = f_i, \quad s(x_{i+1}) = f_{i+1} \\ &s^{(1)}(x_i) = d_i, \quad s^{(1)}(x_{i+1}) = d_{i+1} \\ &s^{(2)}(x_i+) = s^{(2)}(x_i-) \end{aligned}\right\} \quad (2)$$

where, $s^{(1)}(x_i)$ and $s^{(2)}(x_i)$ denotes the first and second order derivative w.r.t x at knot x_i, respectively. The unknowns A_{ij}, j = 0, 1, 2, 3 are given as follows (Karim *et al*., 2016) in Eq. 3:

$$\begin{aligned} &A_{i0} = \alpha_i f_i A_{i1} = (2\alpha_i\beta_i + \alpha_i + \gamma_i) \\ &f_i + \alpha_i h_i d_i, A_{i2} = (2\alpha_i\beta_i + \beta_i + \gamma_i) \\ &f_{i+1} - \beta_i h_i d_{i+1}, A_{i3} = \beta_i f_{i+1} \end{aligned} \quad (3)$$

Applying condition 2, the following tri-diagonal system of linear equations is obtained Eq. 4:

$$a_i d_{i-1} + b_i d_i + c_i d_{i+1} = e_i, \quad i = 1, 2, ..., n-1 \quad (4)$$

With:

$$\begin{aligned} a_i &= h_i\alpha_{i-1}\alpha_i \\ b_i &= h_i\alpha_i(\gamma_{i-1} + 2\alpha_{i-1}\beta_{i-1}) + h_{i-1}\beta_{i-1}(\gamma_i + 2\alpha_i\beta_i) \\ c_i &= h_{i-1}\beta_{i-1}\beta_i \\ e_i &= h_i\alpha_i(\gamma_{i-1} + \alpha_{i-1} + 2\alpha_{i-1}\beta_{i-1})\Delta_{i-1} \\ &+ h_{i-1}\beta_{i-1}(\gamma_i + \beta_i + 2\alpha_i\beta_i)\Delta_i \end{aligned}$$

The system in Eq. 4 gives n-1 linear equations for n+1 unknown derivative values. Thus two more equations are required in order to obtain the unique solution in Eq. 4. The following end points condition i.e. d_0 and d_n can be used (Karim *et al*., 2016) Eq. 5 and 6:

$$s^{(1)}(x_0) = d_0 \quad (5)$$

$$s^{(1)}(x_n) = d_n \quad (6)$$

Both d_0 and d_n is estimated by using arithmetic mean method (Karim *et al*., 2016; Karim, 2017; Sarfraz *et al*., 2005). Note that the system of linear equations given by Eq. 4 is strictly tri-diagonal and has a unique solution for the unknown derivative parameters d_i, i = 1, 2, …, n-1 for all α_i, β_i, $\gamma_i \geq 0$. Thomas's algorithm is used to solve Eq. 4. Thus, $s(x) \in C^2[x_0, x_n]$.

RESULTS AND DISCUSSION

Sufficient condition for constrained interpolation: This study will discusses the constrained data interpolation by using the proposed C^2 rational cubic spline with three parameters. The sufficient condition for the constrained data that lies above arbitrary straight line y = mx+c will be derived. We begin with the problem statement of constrained data interpolation. It stated as follows: given the set of data (x_i, f_i), = 0, 1, …, n lying above the straight line y = mx+c such that Eq. 7:

$$f_i > mx_i + c, i = 0, 1, ..., n \quad (7)$$

Construct the C^2 rational cubic spline interpolant s(x) that lies above the straight line y = mx+c. Mathematically, the curve will lie above the straight y = mx+c, if the C^2 rational cubic spline s(x) defined by Eq. 1 satisfies the following inequality Eq. 8:

$$s(x) > mx + c, \forall x \in [x_0, x_n] \quad (8)$$

Or equivalently Eq. 9:

$$s_i(x) > a_i(1-\theta) + b_i\theta \quad (9)$$

with, $a_i = mx_i+c$, i = 0, 1, …, n and $b_i = mx_{i+1}+c$, i = 0, 1, …, n-1, respectively. Inequality Eq. 9 can be further simplified as Eq. 10:

$$s_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} > a_i(1-\theta) + b_i\theta, i = 0, 1, ..., n-1 \quad (10)$$

Simplify Eq. 10 lead to:

$$s_i(x) = \frac{M_i(\theta)}{Q_i(\theta)} > 0$$

where, Eq. 11:

$$M_i(\theta) = P_i(\theta) - (a_i(1-\theta) + b_i\theta)Q_i(\theta) \quad (11)$$

Eq. 11 is equal to:

$$M_i(\theta) = (1-\theta)^3(f_i - a_i) + (1-\theta)^2\theta M_{i1} + (1-\theta)\theta^2 M_{i2} + \theta^3(f_{i+1} - b_i)$$

With:

$$M_{i1} = f_i + \alpha_i(f_i - a_i) + (f_i - a_i) + h_i d_i - b_i$$

$$M_{i2} = f_{i+1} + \alpha_i(f_{i+1} - b_i) + (f_{i+1} - b_i) - h_i d_{i+1} - a_i$$

Thus, the rational cubic spline interpolant lies above a straight line if and only if the following are satisfied. The necessary conditions: f_i-a_i>0 and f_{i+1}-b_i>0 for i = 0, 1, …, n-1and Eq. 12:

$$M_{i1} > 0 \Rightarrow f_i + \alpha_i(f_i - a_i) + (f_i - a_i) + h_i d_i - b_i > 0 \quad (12)$$

$$M_{i2} > 0 \Rightarrow f_{i+1} + \alpha_i(f_{i+1} - b_i) + (f_{i+1} - b_i) - h_i d_{i+1} - a_i > 0 \quad (13)$$

for i = 0, 1, …, n-1. From Eq. 12 and 13 the following inequalities are obtained Eq. 14:

$$\gamma_i > \frac{\alpha_i(-f_i - h_i d_i + b_i)}{f_i - a_i} \quad (14)$$

and Eq. 15:

$$\gamma_i > \frac{\beta_i(-f_{i+1} + h_i d_{i+1} + a_i)}{f_{i+1} - b_i} \quad (15)$$

Combining Eq. 14 and 15 resulting:

$$\gamma_i > \text{Max}\left\{0, \frac{\alpha_i(-f_i - h_i d_i + b_i)}{f_i - a_i}, \frac{\beta_i(-f_{i+1} + h_i d_{i+1} + a_i)}{f_{i+1} - b_i}\right\}$$

The following theorem state the sufficient condition for C^2 rational cubic spline interpolant for s_i (x) to lies above a straight line.

Theorem 1: The piecewise C^2 rational cubic spline interpolant s (x) defined by Eq. 1 preserves the shape of the data that lies above the straight line y = mx+c, if in each sub-interval [x_i,x_{i+1}], i = 0, 1, …, n-1, the parameters α_i, β_i and γ_i satisfy the following sufficient condition Eq. 16:

$$\alpha_i, \beta_i > 0, \gamma_i > \text{Max}\left\{0, \frac{\alpha_i(-f_i - h_i d_i + b_i)}{f_i - a_i}, \frac{\beta_i(-f_{i+1} + h_i d_{i+1} + a_i)}{f_{i+1} - b_i}\right\} \quad (16)$$

Table 1: Data from (Hussain and Hussain, 2006)

i	x_i	f_i	d_i (C^1)	d_i(C^2)
0	0	20.8	-6.85	-6.85
1	2	12.8	-3.15	-3.15
2	4	10.2	-0.879	-0.880
3	10	12.5	0.585	0.580
4	28	33.9	2.369	2.370
5	30	38.9	2.425	2.400
6	32	43.6	2.275	2.275

Table 2: Data from (Hussain and Hussain, 2006)

i	x_i	f_i	d_i (C^1)	d_i (C^2)
0	12	20.8	-9.10	-9.10
1	4.5	12.8	-5.90	-2.36
2	6.5	10.2	4.50	3.41
3	12	12.5	0.50	0.50
4	7.5	33.9	-3.50	-2.41
5	9.5	38.9	6.90	3.36
6	18	43.6	10.10	10.10

with necessary conditions f_i-a_i>0 and f_{i+1}-b_i>0 for], i = 0, 1, … n-1. The sufficient condition in Eq. 16 can be rewritten as Eq. 17:

$$\alpha_i, \beta_i > 0, \gamma_i = \nu_i + \text{Max}\left\{0, \frac{\alpha_i(-f_i - h_i d_i + b_i)}{f_i - a_i}, \frac{\beta_i(-f_{i+1} + h_i d_{i+1} + a_i)}{f_{i+1} - b_i}\right\}, \nu_i > 0 \quad (17)$$

with 0<ν_i≤0.2. Condition in Eq. 17 will be used to produce C^2 rational cubic interpolant s (x) that lies above a straight line. Case for the data that lies below a straight line can be treated in a same manner as discussed in Karim (2017).

Numerical examples: Three well-known data sets are used to test the capability of the constrained interpolation by using the proposed C^2 rational cubic spline. Mathematica Version 12 is used to produce the numerical and graphical results.

Example 1: A positive data from Hussain and Hussain (2006) is above a straight line y = x+2. Table 1 summarized the value of the first derivatives values d_i both for C^1 and the proposed C^2 rational cubic spline interpolations.

Example 2: Our second example use the data from Hussain and Hussain (2006) lie above a straight line y = x/2+1 as shown in Table 2.

Example 3: The following data lie above a straight line y = 0.5x+0.28. Figure 1-3 show the constrained interpolating curves for data in Tables 1-3, respectively. Figure 1-3a show the default cubic Hermite spline interpolation. Clearly for all data sets, cubic Hermite spline interpolation unable to preserves the shape of

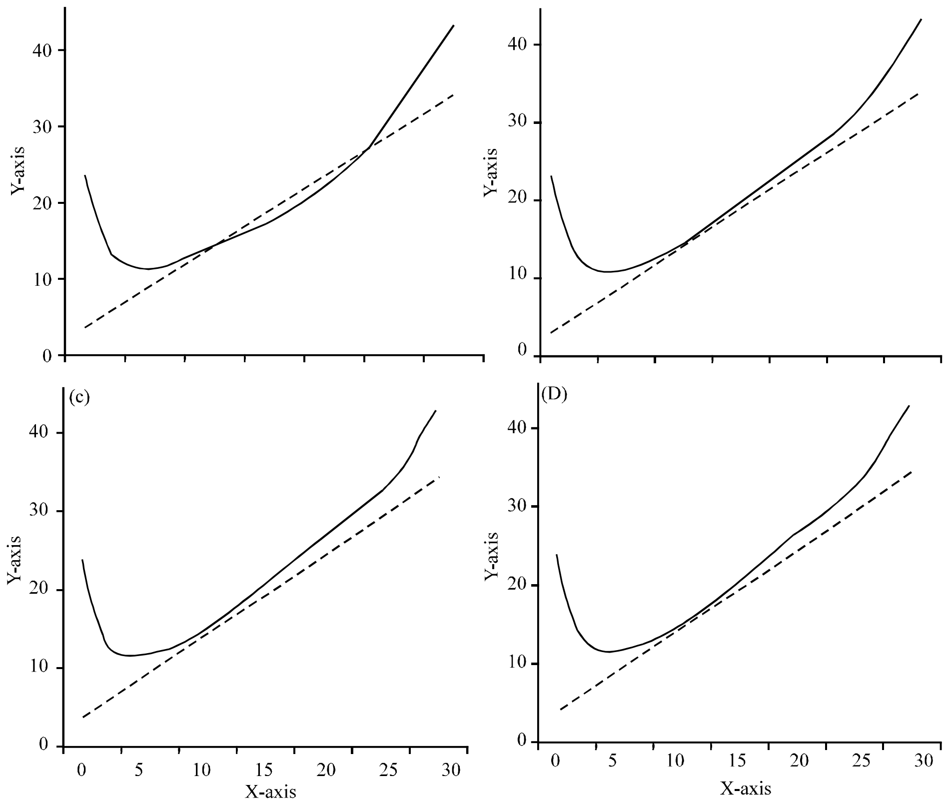

Fig. 1: Comparison with existing schemes: a) Cubic Hermite spline; b) Karim and Kong (2014); c) Proposed scheme with $\alpha_i = \beta_i = 0.5$ and d) Proposed scheme with $\alpha_i = \beta_i = 1$

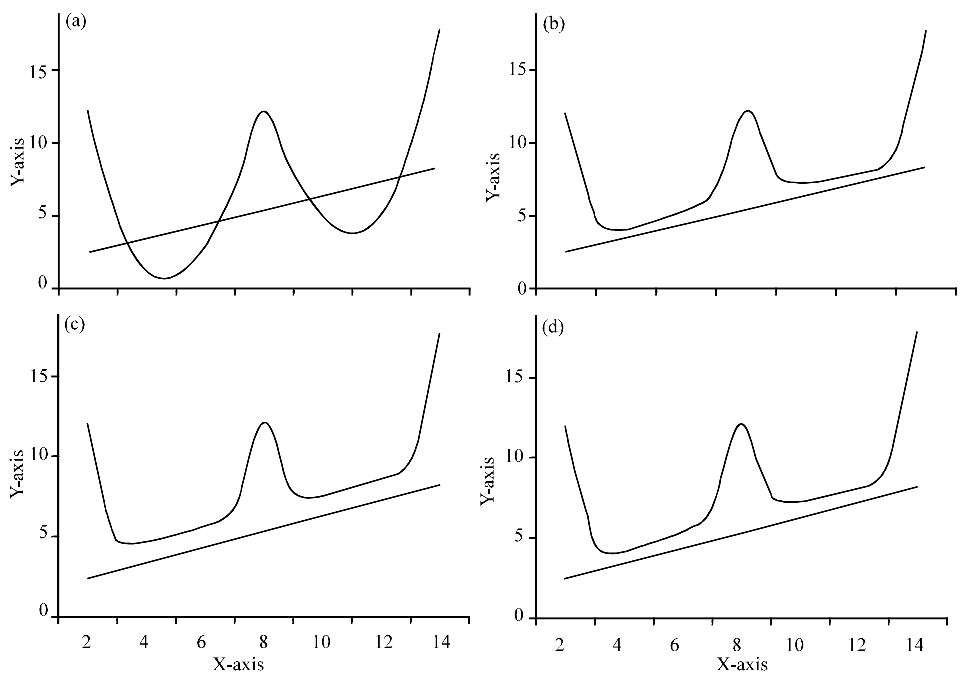

Fig. 2: Comparison with existing schemes: a) Cubic Hermite spline; b) Karim and Kong (2014); c) Proposed scheme with $\alpha_i = \beta_i = 0.5$ and d) Proposed scheme with $\alpha_i = \beta_i = 1$

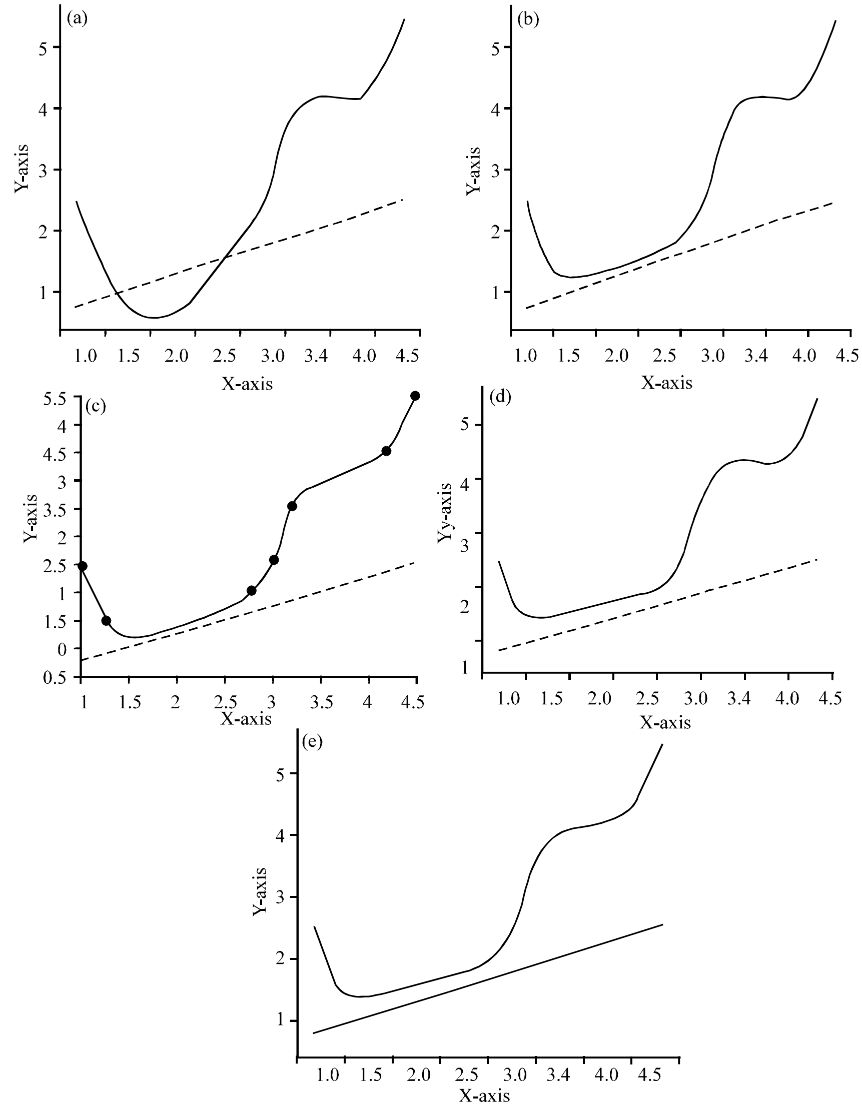

Fig. 3: Comparison interpolating curves: a) Cubic Hermite spline; b) Karim and Kong (2014); c) From Awang *et al.* (2013); d) Proposed scheme with $\alpha_i = \beta_i = 0.25$ and e) Proposed scheme with $\alpha_i = \beta_i = 2.5$

Table 3: Data from (Awang *et al.*, 2013)

i	x_i	f_i	d_i (C^1)	d_i (C^2)
0	1.0	2.5	-4.60	-4.60
1	1.25	1.5	-3.40	-1.532
2	2.8	2.0	2.25	0.857
3	3.0	2.5	3.75	4.468
4	3.2	3.5	4.33	4.489
5	4.2	4.5	2.80	1.970
6	4.5	5.5	3.87	3.87

the data, i.e., there exists some part on the interpolating curve that lies below a straight line. This is unacceptable in the sense of shape prseerving. Figure 1-3b show the shape preserving interpolating curve by using C^1 rational cubic spline proposed by Karim and Kong (2014). Figure 1cd show the proposed C^2 rational cubic spline by varying parameters value $\alpha_i = \beta_i = 0.5$ and $\alpha_i = \beta_i = 1$, respectively. Similarly, Figure 2c, d show the interpolating curve when we apply condition Eq. 17 with different parameter value. Finally Fig. 3c shows the constrained interpolation for data in Table 3 using Awang *et al.* (2013) scheme.

Figure 3c, d in the sub-interval $3.2 \leq x \leq 4.2$, we can see that the proposed scheme give smooth interpolating curve (loose) compare with the research of Awang *et al.* (2013) (more tight). Furthermore the C^2 rational cubic spline give smooth results compare to the C^1 rational cubic spline discussed by Karim and Kong (2014).

CONCLUSION

This study discusses constrained data interpolation using new C^2 rational cubic spline with three parameters of Karim *et al*. (2016). The unknown parameters d_i, i = 1, 2 ,…, n-1 are calculated by utilizing the LU decomposition through Thomas's algorithm with two end point conditions d_0 and d_n are pre-specified (Karim, 2017). The resulting constrained interpolating curve satisfies the C^2 continuity. Free parameters provide extra degree of freedom in controlling the final shape of the interpolating curve while at the same time maintaining C^2 continuity at the respective joint knots. From all numerical results, it can be concluded that the proposed scheme research well and at par with some established schemes. Besides that the proposed scheme gives visual pleasing interpolating curve compare with the research of Awang *et al*. (2013). Finally, research on parametrically shape preserving interpolation is underway. This is very useful for geological spatial interpolation.

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