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Evasion Differential Game with Coordinate-Wise Integral Constraints on Controls of Players

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Abstract: We study an evasion differential game of one evader against one pursuer in the plane $\mathbb{R}^2$ with coordinate-wise integral constraints on the control functions of players. The game is described by some differential equations in terms of each coordinate. The evader moves within a small neighborhood of a vertical η-axis, either by moving vertically or maneuvering. The evader maneuvers to the right, if the pursuer is on its left side and vice versa. We say that evasion is possible, if the position of the evader does not coincide with that of the pursuer at all times. We obtain a sufficient condition of evasion and construct an explicit strategy for the evader to ensure evasion. The strategies depend on the initial positions of players and a defined approached distance between the pursuer and the evader. Each strategy is shown to be admissible by using the fact that the integral constraints are coordinate-wise. By these strategies, evasion is proved to be possible from any initial position of players.

Key words: Control, coordinate-wise integral constraints, differential game, evasion, strategy, maneuvering

INTRODUCTION

Problems of differential game usually involve constructing optimal strategies for players and finding the condition for the objective of the game to be completed. The game involves two opposite parties of players called pursuer and evader whose control functions are usually subjected to the geometric or integral constraints. The evasion differential game is a game in which evader is to avoid being captured by pursuer indefinitely or in a certain time interval.

Among many researches that were devoted to evasion game was by Ibragimov *et al*. (2012a, b). The researchers studied an evasion differential game of one pursuer and one evader with integral constraints, described by the following equation:

$$\dot{z} = A(t)z+B(t)(v-u),\ z(0) = z_0$$

where, $z, u, v\in\mathbb{R}^n$, $z_0\in\mathbb{R}^n/M$ for M is a given closed convex subset of $\mathbb{R}^n$, A(t) and B(t) are continuous n×n matrices and u, v are control parameters of the players. The control functions u(t) and v(t) are subjected to integral constraints. Despite the control resource of pursuer is greater than that of evader, a strategy for evader was constructed to avoid the state of the system to reach the terminal set M.

Differential evasion game with integral constraints on control functions of players could also be studied in the case of many pursuers and one evader. This type of problem was considered by Ibragimov *et al*. (2012a, b) in which the trajectory of players were described by the following equations:

$$P_i : \dot{x}_i = u_i,\ x_i(0) = x_{i0}$$
$$E : \dot{y} = v,\ y(0) = y_0,\ x_{i0} \neq y_0,\ i = 1, ..., m$$

and the control functions of players are subjected to the following conditions:

$$\left(\int_0^\infty |u_i(s)|^2 ds\right)^{\frac{1}{2}} \leq \rho_i,\ \left(\int_0^\infty |v(s)|^2 ds\right)^{\frac{1}{2}} \leq \sigma$$

By assuming that the total resource of the pursuers does not exceed that of the evader, explicit strategy for evader was constructed to obtain the solution of evasion problem in $\mathbb{R}^n$. The evasion game of many pursuers and one evader of integral constraint were further studied by Ibragimov and Salleh (2012) but in the case where the total

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resources of the pursuers are strictly greater than that of the evader. The game occurs in the plane $\mathbb{R}^2$ and despite the pursuers have advantage in total resources it was shown that evasion is possible from some initial positions where the strategy constructed for the evader is based on controls of the pursuers with lag.

Ibragimov *et al.* (2015) continued to investigate an evasion problem of many pursuers against one evader but the space of the game is the Hilbert space ℓ_2. The payoff functional which is the greatest lower bound of distances between the pursuers and evader was considered. The control functions of all players are subjected to integral constraints. It is assumed that the duration of game is fixed and any pursuer's energy is not necessarily greater than the evader's energy. Optimal strategy for the evader was constructed for the evader to maximize the payoff functional.

Another space for an evasion differential game of many pursuers and one evader to occur is manifolds with Euclidean metric. This space was considered in the study by Kuchkarov *et al.* (2016). The motions of all players are simple and players have equal maximal speeds. A method to reduce this game to an equivalent game in $\mathbb{R}^2$ was proposed and a necessary and sufficient condition of evasion was obtained.

In the evasion differential game of one evader against many pursuers studied by Ibragimov *et al.* (2017), the control set of the evader is restricted to a sector S of radius α, $\alpha>1$. The trajectories of players are in $\mathbb{R}^2$ and described by simple differential equation. The sufficient conditions of evasion from any initial positions of players were obtained where the maximum speeds of pursuers are set to be 1 which is less than α.

In the present study, we consider an evasion differential game of one evader from one pursuer with integral constraints on control functions of players. However, we look at the situation where the constraint is coordinate-wise. Evasion occurs in the (ε, η) plane in $\mathbb{R}^2$ where the evader moves within a small neighbourhood of radius ε from a vertical η-axis. A strategy for evader that guarantees evasion from any initial positions of players is constructed.

MATERIALS AND METHODS

Statement of problem: We consider an evasion differential game of one pursuer P and one evader E in the plane $\mathbb{R}^2$ with coordinate-wise integral constraints on controls of players. The game is described by the following differential equations:

$$\begin{aligned} \dot{x}_1 &= u_1, \quad x_1(0) = x_{10} \\ \dot{x}_2 &= u_2, \quad x_2(0) = x_{20} \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{y}_1 &= v_1, \quad y_1(0) = y_{10} \\ \dot{y}_2 &= v_2, \quad y_2(0) = y_{20} \end{aligned} \tag{2}$$

where, $x = (x_1, x_2)$, $y = (y_1, y_2)$, $x_0 = (x_{10}, x_{20})$, $y_0 = (y_{10}, y_{20})$, $x_0 \neq y_0$, $u = (u_1, u_2)$ is control parameter of pursuer, $v = (v_1, v_2)$ is that of evader.

Definition 2.1: A measurable function $u(t) = (u_1, (t), u_2(t))$, $t \geq 0$ is called an admissible control of the pursuer if:

$$\int_0^\infty u_1^2(s)ds \leq \rho_1^2, \int_0^\infty u_2^2(s)ds \leq \rho_2^2 \tag{3}$$

where, ρ_1 and ρ_2 are given positive numbers.

Definition 2.2: A measurable function $v(t) = (v_1(t), v_2(t))$, $t \geq 0$ is called an admissible control of the evader y if:

$$\int_0^\infty v_1^2(s)ds \leq \sigma_1^2, \int_0^\infty v_2^2(s)ds \leq \sigma_2^2 \tag{4}$$

where, σ_1 and σ_2 are given positive numbers.

Definition 2.3: Afunction $V(t, y, x, u)$, $V: (0, \infty)\times \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\rightarrow\mathbb{R}^2$ is called strategy of evader, if for any admissible control of the pursuer, the following initial value problem:

$$\dot{x} = u(t), \ x(0) = x_0$$

$$\dot{y} = V(t, y, x, u(t)), \ y(0) = y_0$$

has a unique absolutely continuous solution $(x(t), y(t))$, $t \geq 0$ and along this solution, the following inequalities hold:

$$\int_0^\infty V_1^2(s, y(s), x(s), u(s))ds \leq \sigma_1^2$$
$$\int_0^\infty V_2^2(s, y(s), x(s), u(s))ds \leq \sigma_2^2$$

Definition 2.4: We say that evasion is possible from the initial positions x_0 and y_0 in the game (Eq. 1-4), if there exists a strategy V of the evader such that for any admissible control of the pursuer, $x(t) \neq y(t)$. In the current research, we construct a strategy for the evader and find conditions for parameters ρ_1, ρ_2, σ_1 and σ_2 that guarantee evasion in game (Eq. 1-4) for any initial positions of players.

RESULTS AND DISCUSSION

Without loss of generality, we assume $t_0 = 0$.

Theorem 3.1: If $\sigma_1>\rho_1$ and $\sigma_2>\rho_2$, then evasion is possible.

Proof: Clearly, the inequalities $\sigma_1>\rho_1$ and $\sigma_2>\rho_2$ imply that $\sigma^2 = \sigma^2_1+\sigma^2_2>\rho^2_1+\rho^2_2 = \rho^2$. Consider two cases: $y_{20}>x_{20}$ or $y_{20}\leq x_{20}$. Let be any number satisfying $0<a<\min\{1, |x_0-y_0|\}$.

Case 1: $y_{20}>x_{20}$ (Fig. 1).

Construction of a strategy for the evader: Define a strategy for the evader as follows:

$$v(t) = \left(0, |u_2(t)|\right),\ t\geq 0 \tag{5}$$

Admissibility of the strategy: Simply:

$$\int_0^\infty v_1^2(s)ds = 0 \leq \sigma_1^2$$

and

$$\int_0^\infty v_2^2(s)ds = \int_0^\infty |u_2(s)|^2\, ds \leq \rho_2^2 \leq \sigma_2^2$$

Hence, strategy (Eq. 5) is admissible.

Proof of evasion is possible: Since, $y_{20}>x_{20}$, we have:

$$y_2(t) = y_{20}+\int_0^t v_2(s)ds = y_{20}+\int_0^t |u_2(s)|ds>x_{20}+$$
$$\int_0^t |u_2(s)|ds \geq x_{20}+\int_0^t u_2(s)ds = x_2(t)$$

Thus, $y(t)\neq x(t)$ for $t\geq 0$ and hence, evasion is possible.

Case 2: $y_{20}\leq x_{20}$ (Fig. 2).

Construction of a strategy for the evader: Define a strategy for the evader as follows. Let:

$$v(t) = \left(0, |u_2(t)|\right) \text{ if } y_2(t)\leq x_2(t) \text{ and } |y(t)\text{-}x(t)|>a \tag{6}$$

At t = 0 by the definition of a, $|y_0-x_0|>a$. If $y_{20}<x_{20}$ then at t = 0, the evader starts to use Eq. 6. Now, let t_1 be the first time when $x_2(t_1) = y_2(t_1)$, $|y(t_1)-x(t_1)|\geq a$. Then the strategy of the evader is as follows:

$$v(t) = \begin{cases} (0,\ \alpha+|u_2(t)|),\ t_1\leq t\leq t_1+\alpha \\ (0,\ |u_2(t)|),\ t>t_1+\alpha \end{cases} \tag{7}$$

where, α is a number that satisfies the following conditions:

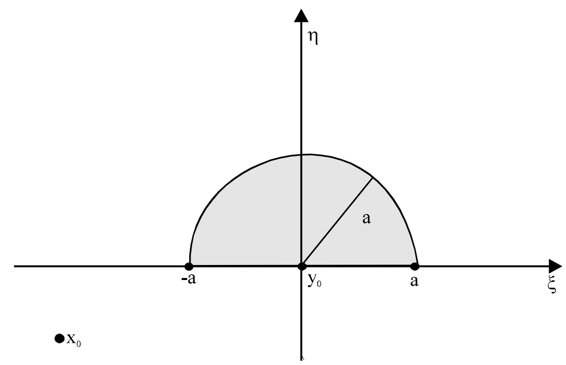

Fig. 1: Positions of players where

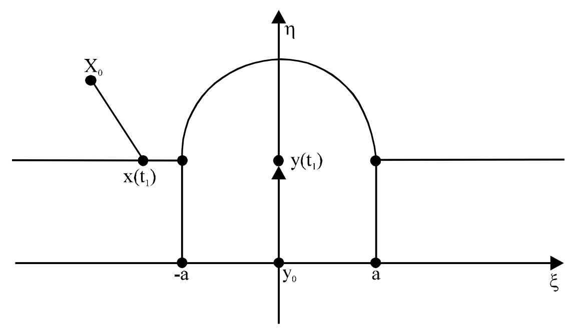

Fig. 2: Positions of players where $y_2(t_1) = x_2(t_1)$

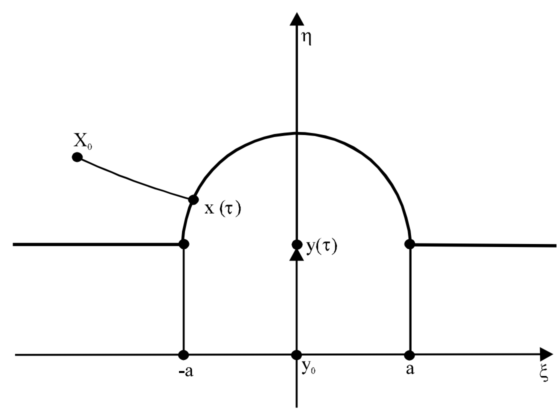

Fig. 3: Positions of players where, $y_2(\tau)<x_2(\tau)$

$$0<\alpha<1,\ a\alpha<\sigma_1^2,\ a\alpha<\sigma_2^2,\ \sqrt{2\alpha}\leq\sigma_1\text{-}\rho_1, \sqrt{2\alpha}\leq\sigma_2\text{-}\rho_2 \tag{8}$$

In particular, if $y_{20} = x_{20}$, then, $t_1 = 0$ and hence, each evader applies strategy (6) on $(0, \infty)$. Note that the time t_1 may not occur. In this case, clearly evasion is possible. Let now the second inequality in Eq. 6 fails to hold first time at some $t = \tau$ that is $|y(\tau)-x(\tau)| = a$ but for $0\leq t<\tau$, $|y(t)-x(t)|>a$ and $x_2(t)>y_2(t)$ as illustrated in Fig. 3. Then, we set:

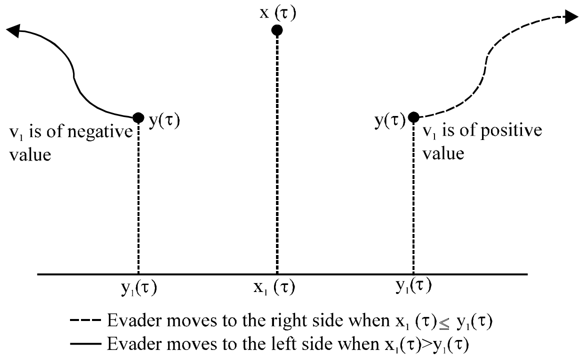

Fig. 4: Strategy of players

$$v(t)=\begin{cases}\pm(\alpha+|u_1(t)|),\ \alpha+|u_2(t)|,\ \tau\le t\le\tau',\\ (0,\ \alpha+|u_2(t)|),\ \tau'<t\le\tau'+\dfrac{a}{\alpha},\\ (0,|u_2(t)|),\ t>\tau'+\dfrac{a}{\alpha}\end{cases}\qquad(9)$$

Moreover, ± in Eq. 9 means:

$$v_1(t)=\begin{cases}\alpha+|u_1(t)|,\ x_1(\tau)\le y_1(\tau)\\ -(\alpha+|u_1(t)|),\ x_1(\tau)>y_1(\tau)\end{cases}\qquad(10)$$

that is if the position of the pursuer is on the left side of the evader then the evader will maneuver to the right side to avoid from the pursuer and vice versa (Fig. 4).

Admissibility of the strategy: We now show admissibility of the strategy of the evader. For strategies, Eq. 6 and 7 it is clear that:

$$\int_0^\infty v_1^2(s)ds=0\le\sigma_1^2$$

Now, for strategy, Eq. 6:

$$\int_0^\infty v_1^2(s)ds=\int_0^\infty|u_2(s)|^2ds\le\rho_2^2\le\sigma_2^2$$

If time t_1 occurs, then by Eq. 7 and the fifth inequality in Eq. 8, we have:

$$\int_0^\infty v_2^2(s)ds=\int_0^{t_1}u_2(s)|^2|\,ds+\int_{t_1}^{t_1+\alpha}(\alpha+|u_2(s)|^2ds+\int_{t_1+\alpha}^\infty|u_2(s)|^2\,ds=\int_{t_1}^{t_1+\alpha}\alpha^2ds+2\alpha\int_{t_1}^{t_1+\alpha}|u_2(s)|\,ds+\int_0^\infty|u_2(s)|^2\,ds\le\alpha^3+2\rho_2\sqrt{\alpha^3}+\rho_2^2\le\alpha+2\rho_2\sqrt{\alpha}+\rho_2^2=(\sqrt{\alpha}+\rho_2)^2\le(\sqrt{2\alpha}+\rho_2)^2\le\sigma_2^2$$

We conclude that the strategy is admissible. For strategy Eq. 9 by the 4th inequality in Eq. 8, we get:

$$\int_0^\infty v_1^2(s)ds=\int_0^\tau 0ds+\int_\tau^{\tau'}(\alpha+|u_1(s)|)^2ds+\int_{\tau'}^{\tau'+\frac{a}{\alpha}}0ds+\int_{\tau'+\frac{a}{\alpha}}^\infty 0ds\le a\alpha+2\rho_1\sqrt{a\alpha}+\rho_1^2\le\alpha+2\rho_1\sqrt{\alpha}+\rho_1^2=(\sqrt{\alpha}+\rho_1)^2\le(\sqrt{2\alpha}+\rho_1)^2\le\sigma_1^2$$

Also by the fifth inequality in Eq. 8, we have:

$$\int_0^\infty v_2^2(a)ds=\int_0^\tau|u_2(s)|^2\,ds+\int_\tau^{\tau'}(\alpha+|u_2(s)|)^2ds+\int_{\tau'}^{\tau'+\frac{a}{\alpha}}(\alpha+|u_2(s)|)^2ds+\int_{\tau'+\frac{a}{\alpha}}^\infty|u_2(s)|^2\,ds\le 2a\alpha+2\rho_2\sqrt{2a\alpha}+\rho_2^2\le 2\alpha+2\rho_2\sqrt{2\alpha}+\rho_2^2=(\sqrt{2\alpha}+\rho_2)^2\le\sigma_2^2$$

The proof is thus, completed.

Proof of evasion is possible: While the evader applies Eq. 6, it is clear that $y(t)\ne x(t)$, since, $|y(t)-x(t)|>a>0$. If time t_1 occurs then $y(t)\ne x(t)$, $t\ge t_1$. Indeed, since, $|y(t_1)-x(t_1)|\ge a>0$, we have $y(t_1)\ne x(t_1)$. For $t>t_1$, we have $y_2(t)>x_2(t)$ which implies $y(t)\ne x(t)$ where the proof is similar to that of case 1. Let time τ occurs. Then the evader uses strategy Eq. 9. We estimate the distance between players as follows. For time $t\in[\tau,\tau']$:

$$|y(t)-x(t)|=\left|\left(y(\tau)+\int_\tau^t v(s)ds\right)-\left(x(\tau)+\int_\tau^t u(s)ds\right)\right|\ge\left|y(\tau)-x(\tau)\right|-\left|\int_\tau^t v(s)ds-\int_\tau^t u(d)ds\right|\ge\left|y(\tau)-x(\tau)\right|-\left|\int_\tau^t v(s)ds\right|-\left|\int_\tau^t u(s)ds\right|\ge a-\int_\tau^t|v(s)|\,ds-\int_\tau^t|u(s)|\,ds\ge a-\sigma\sqrt{t-\tau}-\rho\sqrt{t-\tau}>a-2\sigma\sqrt{t-\tau}$$

On the other hand, without loss of generality by assuming $x_1(\tau)\le y_1(\tau)$ by Eq. 10, we have:

$$|y(t)-x(t)|\ge|y_1(t)-x_1(t)|=\left|\left(y_1(\tau)+\int_\tau^t v_1(s)ds\right)-\left(x_1(\tau)+\int_\tau^t u_1(s)ds\right)\right|\ge|y_1(\tau)-x_1(\tau)|+\int_\tau^t(\alpha+|u_1(s)|)ds-\left|\int_\tau^t u_1(s)ds\right|\ge|y_1(\tau)-x_1(\tau)|+\int_\tau^t\left(\alpha+|u_1(s)|-|u_1(s)|\right)ds\ge\int_\tau^t\alpha ds=\alpha(t-\tau)$$

Now, let $z_1(t)=a-2\sigma\sqrt{t-\tau}$ and $z_2(t)=a(t-\tau)$ for $t\in[\tau-\tau']$. As t progress from τ to τ', we have $0\le t-\tau\le a/\alpha$ which implies $a\ge z_1(t)\ge a-2\sigma\sqrt{a/\alpha}$ that is $z_1(t)$ decreases

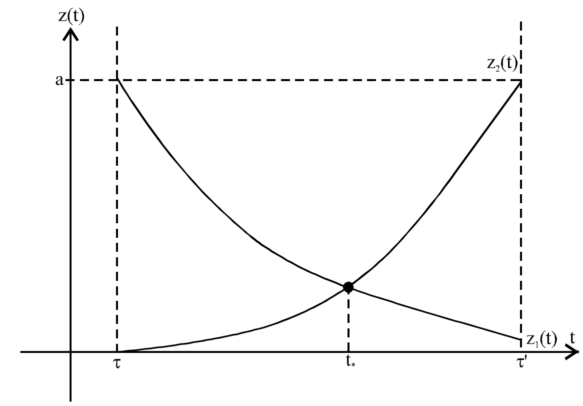

Fig. 5: Function

continuously from a to $a\text{-}2\sigma\sqrt{a/\alpha}$ on [τ, τ']. On the other hand, the inequality $0\leq t\text{-}\tau\leq a/\alpha$ implies $0\leq z_2(t)\leq a$ that is $z_2(t)$ increases continuously from 0 to a on [τ, τ′]. Hence, the function z(t) has a unique minimum point on [τ, τ′] at some t = t when $z_1(t) = z_2(t)$ (Fig. 5). By a straight forward calculation, we can see that:

$$t_* = \tau+\frac{a^2}{(\sigma+\sqrt{\sigma^2+a\alpha})^2}\in[\tau, \tau']$$

Thus:

$$z_1(t)>\frac{a\left(\sqrt{\sigma^2+a\alpha}\text{-}\sigma\right)}{\sigma\left(\sqrt{3/2}+1\right)}>\frac{a^2\alpha}{\sigma^2\left(\sqrt{3/2}+1\right)^2}>\frac{4a^2\alpha}{25\sigma^2}>0$$

And:

$$z_2(t)>\frac{a^2\alpha}{\left(\sqrt{3/2\,\sigma^2}+\sigma\right)^2}=\frac{a^2\alpha}{\sigma^2\left(\sqrt{3/2}+1\right)}>\frac{4a^2\alpha}{25\sigma^2}>0$$

So that:

$$|\,y(t)\text{-}x(t)\,|\geq \max\{z_1(t), z_2(t)\}>\frac{4a^2\alpha}{25\sigma^2}>0,\ t\in[\tau, \tau']$$

In particular:

$$|\,y(\tau')\text{-}x(\tau')\,|>\frac{4a^2\alpha}{25\sigma^2}>0$$

Hence, y (τ′)≠x (τ′). Now:

$$y_2(\tau')\text{-}x_2(\tau')=\left(y_2(\tau)+\int_\tau^{\tau'}v_2(s)ds\right)\text{-}$$

$$\left(x_2(\tau)+\int_\tau^{\tau'}u_2(s)ds\right)=\left(y_2(\tau)\text{-}x_2(\tau)\right)+$$

$$\int_\tau^{\tau'}\left(\alpha+|u_2(s)|ds\right)\text{-}\int_\tau^{\tau'}u_2(s)ds\geq y_2(\tau)\text{-}$$

$$x_2(\tau)+\int_\tau^{\tau'}\alpha ds\geq \text{-}a+\alpha(\tau'\text{-}\tau)=0$$

Thus, y_2 (τ′)≥x_2 (τ′). This means that the position of the pursuer cannot be above the horizontal line K = y_2(τ′) in the plane $\mathbb{R}^2$ at t = τ′. Hence:

$$y_2(t)\text{-}x_2(t)=\left(y_2(\tau')+\int_{\tau'}^t v_2(s)ds\right)\text{-}$$

$$\left(x_2(\tau')+\int_{\tau'}^t u_2(s)ds\right)=\left(y_2(\tau')\text{-}x_2(\tau')\right)+$$

$$\int_{\tau'}^t\left(\alpha+|u_2(s)|\right)ds\text{-}\int_{\tau'}^t u_2(s)ds\geq y_2(\tau')\text{-}x_2(\tau')+$$

$$\int_{\tau'}^t\alpha ds\geq\alpha(t\text{-}\tau')>0$$

Therefore, y(t)≠x(t) for t∈(τ′, τ′+a/α). In particular:

$$y_2\left(\tau'+\frac{a}{\alpha}\right)\text{-}x_2\left(\tau'+\frac{a}{\alpha}\right)>0 \tag{11}$$

Using Eq. 11 for t∈(τ′+a/α, ∞), we have:

$$y_2(t)-x_2(t)=y_2\left(\tau'+\frac{a}{\alpha}\right)-x_2\left(\tau'+\frac{a}{\alpha}\right)+$$

$$\int_{\tau'+\frac{a}{\alpha}}^t v_2(s)ds-\int_{\tau'+\frac{a}{\alpha}}^t u_2(s)ds=$$

$$y_2\left(\tau'+\frac{a}{\alpha}\right)\text{-}x_2\left(\tau'+\frac{a}{\alpha}\right)+\int_{\tau'+\frac{a}{\alpha}}^t u_2(s)ds\text{-}$$

$$\int_{\tau'+\frac{a}{\alpha}}^t u_2(s)ds\geq y_2\left(\tau'+\frac{a}{\alpha}\right)\text{-}x_2\left(\tau'+\frac{a}{\alpha}\right)>0$$

Hence, y(t)≠x(t) for t∈(τ′+a/α, ∞). We conclude that, y(t)≠x (t), t≥0 and proof is completed.

CONCLUSION

The contribution of the present research is a study of an evasion differential game of one evader from one pursuer with coordinate-wise integral constraints on control functions of the players. The sufficient conditions for evasion is possible are $\sigma_1>\rho_1$ and $\sigma_2>\rho_2$ and with the constructed strategy it is shown that evasion is possible from every possible location of initial position of players.

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