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Modelings of Microeconomic Structures Using a Probablistic Approche

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Abstract: Given m Microeconomic Structures (MS) and n individuals, on the basis of their will, a random or conditional assignment, we will give the probability laws characterizing the distributions of these individuals in those structures also we will use the definition of the factorial moment, the number of stirling and some classic results that link the factorial moment to the high order moment.

Key words: Microeconomic structures, Faa Di Bruno's formula, high order moments, probability distributions, factorial moments, conditional assignment

INTRODUCTION

We first recall the results found in the study (Mohammed *et al*., 2016) including the new formula of Faa Di Bruno:

$$\forall m \geq 1$$

$$(fog)^{(m)} = \sum_{p=1}^{m} \left(f^{(p)} og\right)\left\{\sum_{\sum_{i=1}^{p} a_i = m-p} \prod_{i=1}^{p} C_{m-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} g^{(1+a_i)}\right\} \quad (1)$$

With:

$$\sigma(a_i) = \begin{cases} 1 & \text{if } i = 1 \\ i + \sum_{j<i} a_j & \text{if } i \geq 2 \end{cases}$$

Definition 1: For all r∈R the r-th factorial moment of X defined by Johnson *et al*. (2005), Scheaffer and Young (2010) is given by the formula:

$$E\left[(X)_r\right] = E\left[X(X-1)(X-2)\ldots(X-r+1)\right] \quad (2)$$

With $(x)_r$ a special operator defined as:

$$(x)_r = x(x-1)(x-2)\ldots(x-r+1)$$

We also have in Johnson *et al*. (2005):

$$E\left[(X)_r\right] = g_x^{(r)}(1) \quad (3)$$

With g_x the generating function (DasGupta, 2010; Stirzaker, 1999; Gallier, 2017) of the random variable X (Forbes *et al*., 2010; Hsu, 1997; Gane, 2016) defined in Johnson *et al*. (2005) as follows:

$$\forall i \in N,\ P[X = i] = \frac{g_x^{(i)}(0)}{i!} \quad (4)$$

Definition 2: Let, L = {B_1, B_2, ..., B_p} be a set of p subsets of a set E. L is a partition of E into p classes if:

- None of the parties B_k is empty
- The parties B_k are disjoint in pairs
- The reunion of the parties B_k is equal to the set E

We note $\left\{{n \atop p}\right\}$ the number of partitions into p classes of a set with n elements. The coefficients $\left\{{n \atop p}\right\}$ are called Stirling numbers of the second kind (Farhi, 2014; Gould, 1964) that verifies:

- $\forall n \in N^*$, $\left\{{n \atop 1}\right\} = 1$, $\left\{{n \atop n}\right\} = 1$, $\left\{{n \atop p}\right\} = 0$ if $p>n$
- $\forall n \geq 2$ we have: $\left\{{n \atop n-1}\right\} = \binom{n}{2}$ and $\left\{{n \atop 2}\right\} = 2^{n-1}-1$
- $\forall 1 \leq p<n$ we also have: $\left\{{n \atop p}\right\} = \left\{{n-1 \atop p-1}\right\} + p\left\{{n-1 \atop p}\right\}$

∀x∈ℝ. ∀n∈N. with $x^{\underline{0}} = 1$ and $x^{\underline{n+1}} = (x-n)x^{\underline{n}}$, then we have:

$$x^n = \sum_{k=0}^{n} \left\{{n \atop k}\right\} x^{\underline{k}} \quad (5)$$

MATERIALS AND METHODS

Recursive formula for calculating higher order moments through factorial moments

Case of classic series

Definition 3: Let, $g_{x_1}, g_{x_2} \ldots g_{x_n}$ be the generating functions of n independent and identical random variables (DasGupta, 2010; Forbes *et al*., 2010) X_1, X_2 ... X_n with values in N. The random variable $S_n = X_1+X_2+ \ldots X_n$ has a generating function represented as:

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$$g_{S_n}(t) = (g_x(t))^n \tag{6}$$

Proposition 1: The m-th factorial moment of the random variable S_n can be written as:

$$E\left[(S_n)_m\right] = g_{S_n}^{(m)}(1) = \sum_{\substack{\sum_{i=1}^m i a_i = m \\ \sum_{i=1}^m a_i = p}} A_n^p \frac{m!}{\prod_{i=1}^m a_i!(i!)^{a_i}} \prod_{i=1}^m \left(E\left[(X)_i\right]\right)^{a_i}$$

$$E\left[(S_n)_m\right] = \sum_{p=1}^m A_n^p \left\{\sum_{\sum_{i=1}^p a_i = m-p} \prod_{i=1}^p C_{m-\sigma(a_i)}^{a_i} \prod_{i=1}^p \left(E\left[(X)_{1+a_i}\right]\right)\right\}$$

Proof 1: According to Faa Di Bruno's formula, the m-th derivative of a composite function given in Mohammed *et al*. (2016) and Johnson (2002) by:

$$(fog)^{(m)} = \sum_{\substack{\sum_{i=1}^m i a_i = m \\ \sum_{i=1}^m a_i = p}} \left(f^{(p)}og\right) \frac{m!}{\prod_{i=1}^m a_i!(i!)^{a_i}} \prod_{i=1}^m \left(g^{(i)}\right)^{a_i} \tag{7}$$

We put $f(t) = t^n$ in order to have $g_{S_n}(t) = (fog_x)(t)$. Then, we get:

$$g_{S_n}^{(m)}(t) = (fog_x)^{(m)}(t)$$

By applying the formula of Faa Di Bruno with $f^{(p)}(t) = A_n^p t^{n-p}$ and $g_x(1) = 1$, we get:

$$(fog_x)^{(m)}(t) = \sum_{\substack{\sum_{i=1}^m i a_i = m \\ \sum_{i=1}^m a_i = p}} \left(f^{(p)}og_x\right)(t) \frac{m!}{\prod_{i=1}^m a_i!(i!)^{a_i}} \prod_{i=1}^m \left(g_x^{(i)}(t)\right)^{a_i}$$

By choosing t = 1, we obtain the following equation:

$$(fog_x)^{(m)}(1) = E\left[(S_n)_m\right] = \sum_{\substack{\sum_{i=1}^m i a_i = m \\ \sum_{i=1}^m a_i = p}} A_n^p \frac{m!}{\prod_{i=1}^m a_i!(i!)^{a_i}} \prod_{i=1}^m \left(E\left[(X)_i\right]\right)^{a_i}$$

The second expression of $E[(S_n)_m]$ can be found by following exactly the same steps of the proof bellow, using the formula of Faa Di Bruno in Eq. 1.

This theorem allows us to calculate the factorial moments of the random variable S_n directly. More than this based on this theorem we establish a simple formula that will give us directly the higher order moments (Johnson *et al*., 2005; Scheaffer and Young, 2010) of the random variable S_n.

Theorem 1: The m-th moment of the random variable S_n is given by:

$$E\left[(S_n)^m\right] = \sum_{k=0}^m \left\{ {m \atop k} \right\} E\left[(S_n)_k\right] = \sum_{k=0}^m \left\{ {m \atop k} \right\} \sum_{\substack{\sum_{i=1}^k i a_i = k \\ \sum_{i=1}^k a_i = p}} \frac{n!}{(n-p)!} \frac{k!}{\prod_{i=1}^k a_i!(i!)^{a_i}} \prod_{i=1}^k \left(E\left[(X)_i\right]\right)^{a_i}$$

$$E\left[(S_n)^m\right] = \sum_{k=0}^m \left\{ {m \atop k} \right\} \sum_{p=1}^k A_n^p \left\{\sum_{\sum_{i=1}^p a_i = k-p} \prod_{i=1}^p C_{k-\sigma(a_i)}^{a_i} \prod_{i=1}^p \left(E\left[(X)_{1+a_i}\right]\right)\right\}$$

Proof 2: According to the previous theorem, we have:

$$E\left[(S_n)_m\right] = \sum_{\substack{\sum_{i=1}^m i a_i = m \\ \sum_{i=1}^m a_i = p}} A_n^p \frac{m!}{\prod_{i=1}^m a_i!(i!)^{a_i}} \prod_{i=1}^m \left(E\left[(X)_i\right]\right)^{a_i}$$

We have mentioned before that $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$ with $x^{\underline{0}} = 1$ and $x^{\underline{n+1}} = (x-n)x^{\underline{n}}$:

$$x^n = \sum_{k=0}^n \left\{ {n \atop k} \right\} x^{\underline{k}}$$

With the linearity of the expectation (Gallier, 2017; Pitman, 1997), we find the desired result. The second expression of $E[(S_n)^m]$ can be found by following exactly the same steps of this proof (2) using the formula of Faa Di Bruno mentioned above in Eq. 1.

Verification: Let us calculate $E[S_n^2]-E[S_n]^2$ (Stirzaker, 1999; Gallier, 2017) using the theorem 1:

$$E\left[S_n^2\right] - E\left[S_n\right]^2 = \sum_{k=0}^2 \left\{ {2 \atop k} \right\} E\left[(S_n)_k\right] - \left(\sum_{k=0}^1 \left\{ {1 \atop k} \right\} E\left[(S_n)_k\right]\right)^2$$

Because:

$$\left\{ {n \atop 1} \right\} = 1, \left\{ {n \atop n} \right\} = 1 \text{ and } \left\{ {n \atop 0} \right\} = 0$$

$$\begin{aligned} E\left[S_n^2\right] - E\left[S_n\right]^2 &= E\left[(S_n)_1\right] + E\left[(S_n)_2\right] - \left(E\left[(S_n)_1\right]\right)^2 \\ &= E\left[S_n\right] + E\left[S_n(S_n-1)\right] - E\left[S_n\right]^2 \\ &= E\left[S_n^2\right] - E\left[S_n\right]^2 \\ &= \sigma_{S_n}^2 \end{aligned}$$

Lemma 1: If X is a random variable then the high order moments and the factorial moments of X can be linked by Eq. 8:

$$E\left[X^n\right] = \sum_{k=0}^n \left\{ {n \atop k} \right\} E\left[(X)_k\right] \tag{8}$$

Theorem 2: Let, $X_1, X_2, ..., X_n$ be an independant and identical random variables with values in N, for $S_n = \sum_{i=1}^{n} X_i$ we have:

$$P[S_n = m] = \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} A_n^p P[X=0]^{n-p} \prod_{i=1}^{m} \frac{(P[X=i])^{a_i}}{a_i!}$$

$$P[S_n = m] = \sum_{p=1}^{m} \frac{A_n^p}{m!} P[X=0]^{n-p} \times \left\{ \sum_{\sum_{i=1}^{p} a_i = m-p} \prod_{i=1}^{p} A_{m-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} (1+a_i) P[X = 1+a_i] \right\}$$

Proof 3: We apply the formula of Faa Di Bruno for $f(t) = t^n$ and $g_x(0) = P[X = 0]$ which leads to:

$$g_{S_n}^{(m)}(t) = (f o g_x)^{(m)}(t) = \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} \left(f^{(p)} o g_x\right)(t) \frac{m!}{\prod_{i=1}^{m} a_i!(i!)^{a_i}} \prod_{i=1}^{m} \left(g_x^{(i)}(t)\right)^{a_i}$$

By choosing t = 0, we find:

$$g_{S_n}^{(m)}(0) = \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} A_n^p (P[X=0])^{n-p} \frac{m!}{\prod_{i=1}^{m} a_i!(i!)^{a_i}} \prod_{i=1}^{m} (i! P[X=i])^{a_i}$$

$$= \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} A_n^p (P[X=0])^{n-p} \frac{m!}{\prod_{i=1}^{m} a_i!} \prod_{i=1}^{m} (P[X=i])^{a_i}$$

$$= \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} A_n^p (P[X=0])^{n-p} m! \prod_{i=1}^{m} \frac{(P[X=i])^{a_i}}{a_i!}$$

Then, we use Eq. 4:

$$g_{S_n}^{(m)}(0) = m! P[S_n = m]$$

The second expression of P $[S_n = m]$ can be found by following exactly the same steps of the proof (3), using the formula of Faa Di Bruno in Eq. 1.

Case of random series: Let N be a random variable in N and $X_1, X_2, ..., X_N$ an independent identical random variables with values also in N. The random variable $S_N = \sum_{i=1}^{N} X_i$ has as a generating function:

$$g_{S_N}(t) = (g_N o g_X)(t) \tag{9}$$

Theorem 3: The m-th factorial moment this time is given by:

$$E[(S_N)_m] = \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} g_N^{(p)}(1) \frac{m!}{\prod_{i=1}^{m} a_i!(i!)^{a_i}} \prod_{i=1}^{m} \left(E[(X)_i]\right)^{a_i}$$

$$E[(S_N)_m] = \sum_{p=1}^{m} g_N^{(p)}(1) \left\{ \sum_{\sum_{i=1}^{p} a_i = m-p} \prod_{i=1}^{p} C_{m-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} E\left[(X)_{1+a_i}\right] \right\}$$

The m-th moment of the random variable S_N can be written as:

$$E\left[(S_N)^m\right] = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} E\left[(S_N)_k\right] = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} \sum_{\substack{\sum_{i=1}^{k} i a_i = k \\ \sum_{i=1}^{k} a_i = p}} g_N^{(p)}(1) \frac{k!}{\prod_{i=1}^{k} a_i!(i!)^{a_i}} \prod_{i=1}^{k} \left(E[(X)_i]\right)^{a_i}$$

$$E\left[(S_N)^m\right] = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} \times \sum_{p=1}^{k} g_N^{(p)}(1) \left\{ \sum_{\sum_{i=1}^{p} a_i = k-p} \prod_{i=1}^{p} C_{k-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} \left(E\left[(X)_{1+a_i}\right]\right) \right\}$$

The probability law is represented as:

$$P[S_N = m] = \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} g_N^{(p)}(P[X=0]) \prod_{i=1}^{m} \frac{(P(X=i))^{a_i}}{a_i!}$$

$$P[S_N = m] = \frac{1}{m!} \sum_{p=1}^{m} g_N^{(p)}(P[X=0]) \times \left\{ \sum_{\sum_{i=1}^{p} a_i = m-p} \prod_{i=1}^{p} A_{m-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} (1+a_i) P[X = 1+a_i] \right\}$$

Proof 4: We have $E[(S_N)_m] = (g_N o g_X)^{(m)}(1)$, if we apply the formula of Faa Di Bruno, then

$$(g_N o g_X)^{(m)}(t) = \sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} g_N^{(p)} o g_X(t) \frac{m!}{\prod_{i=1}^{m} a_i!(i!)^{a_i}} \prod_{i=1}^{m} \left(g_X^{(i)}(t)\right)^{a_i}$$

By choosing t = 1 we end the proof.

The second expression of $E[(S_N)_m]$ can be found by following exactly the same steps of the proof (4) using the second version of Faa Di Bruno Eq. 1. We have $g_{S_N}^{(m)}(t) = (g_N o g_X)^{(m)}(t)$. By taking t = 0, we get:

$$(g_N o g_X)^{(m)}(0) =$$
$$\sum_{\substack{\Sigma_{i=1}^{m} i a_i = m \\ \Sigma_{i=1}^{m} a_i = p}} \frac{g_N^{(p)}\left(P[X=0]\right) m!}{\prod_{i=1}^{m} a_i!(i!)^{a_i}} \prod_{i=1}^{m}\left(i! P[X=i]\right)^{a_i}$$
$$= \sum_{\substack{\Sigma_{i=1}^{m} i a_i = m \\ \Sigma_{i=1}^{m} a_i = p}} g_N^{(p)}\left(P[X=0]\right) \frac{m!}{\prod_{i=1}^{m} a_i!} \prod_{i=1}^{m}\left(P[X=i]\right)^{a_i}$$

The second expression of P $[S_N = m]$ can be found by following exactly the same steps, using the formula of Faa Di Bruno mentioned above in Eq. 1.

Application: We consider (Forbes *et al*., 2010; Walk, 2007; Zelterman, 2005):
$X_i \rightsquigarrow$ Binomial (l, p) and $N \rightsquigarrow$ Bernoulli. It is evident that:

$$g_X(t) = \sum t^n P[X=n] = \sum t^n C_l^n p^n q^{l-n} =$$
$$\sum C_l^n (pt)^n q^{l-n} = (pt+q)^l$$

And:

$$g_N(t) = \sum t^n P[X=n] = q+pt$$

Then, we write:

$$g_{S_N}(t) = (g_N o g_X)(t) = q+p(pt+q)^l =$$
$$q+p\sum_{k=0}^{l} C_l^k (pt)^k q^{l-k} = \sum_{k=0}^{l} d_k t^k + q$$

Where:

$$\begin{cases} d_0 = pq^l \\ d_k = C_l^k p^{k+1} q^{l-k} \\ d_l = p^{l+1} \end{cases}$$

With $g_{S_N}^{(n)}(0) = n! d_n$ we obtain the following equation:

$$P[S_N = n] = d_n =$$
$$\sum_{\substack{\Sigma_{i=1}^{n} i a_i = n \\ \Sigma_{i=1}^{n} a_i = p}} g_N^{(p)}\left(P[X=0]\right) \prod_{i=1}^{n} \frac{P[X=i]^{a_i}}{a_i!} \quad (10)$$

RESULTS AND DISCUSSION

Combinatorics linking a set of microeconomic structures: Suppose that we have n individuals that we distribute in m microeconomic structures (Hall and Lieberman, 2009; Nicholson and Snyder, 2007) such as:

$$\begin{cases} m_1 \text{ of type } t_1 \\ m_2 \text{ of type } t_2 \\ \quad . \\ \quad . \\ \quad . \\ m_d \text{ of type } t_d \end{cases}$$

With m_i the number of the microeconomic structure of type t_i that verifies $\sum_{i=1}^{d} m_i = m$. Let:

X_i be the random variable that denotes the number of the individuals in all the microeconomic structures of type t_i. $X = (X_1, X_2, ..., X_d)$ the random vector defined from the random variables X_i. A_m defines the sets of the microeconomic structures and A_i a part of A_m containing all the microeconomic structures of type t_i. Therefore:

$$A_m = U_{i=1}^{d} A_i$$

A disjoint meeting. Moreover, X_i may be presented as:

$$X_i : P(A_m) \to N \quad A_i \to X_i(A_i) = x_i$$

It can also be made to say that:

$$[X_i = x_i] = X_i^{-1}(x_i) = A_i \quad (11)$$

which implies that:

$$[X_1 = x_1 \ldots X_d = x_d] =$$
$$X_1^{-1}(x_1) \times \ldots \times X_d^{-1}(x_d) = A_1 \times \ldots \times A_d$$

To simplify the calculation, we put:

$$A_{1,\ldots,d} = [X_1 = x_1 \ldots X_d = x_d] \quad (12)$$

Also, we note:

$$\tilde{A}_1 = [X_i \le x_i] = U_{k_i=0}^{x_i} [X_i = k_i] \quad (13)$$

A disjoint meeting, then:

$$P(\tilde{A}_1) = \sum_{k_i=0}^{x_i} P[X_i = k_i]$$

The same way, we note:

$$\tilde{A}_{1,\ldots,d} = [X_1 \le x_1 \ldots X_d \le x_d] =$$
$$U_{\sum_{i=1}^{d} k_i = 0}^{\sum_{i=1}^{d} x_i} [X_1 = k_1 \ldots X_d = k_d] \quad (14)$$

So:

$$P(\tilde{A}_{1,\ldots,d}) = \sum_{\sum_{i=1}^{d} k_i = 0}^{\sum_{i=1}^{d} x_i} P[X_1 = k_1 \ldots X_d = k_d]$$

Our goal is to calculate the probability distribution of $A_{1,\ldots,d}$ A_i $\tilde{A}_i$ and $\tilde{A}_{1,\ldots,d}$ in the following three possible distributions:

Distribution of n individuals at random in m microeconomic structures. Distribution of n individuals numbered from 1 to n in m microeconomic structures.

Distributions of n individuals in m microeconomic structures knowing that each one can contain only one individual at most.

Random distribution of n individuals in m microeconomic structures: If we randomly distribute n individuals in m microeconomic structures, two distributions differ only by the number of the individuals in at least one n structure. Since, the affectivity of the individuals is random, then we can have several individuals in single microeconomic structures as we can have empty ones. The following lemma makes it possible to calculate the number of ways to place n individuals in m microeconomic structures at random.

Lemma 2: Let Ω be the universe made up of ways to place n individuals in m microeconomic structures, its cardinal is the number of ways to write the integer n as the sum of m natural integers. It is equal to:

$$\text{card}\,\Omega = C_{n+m-1}^{n} \quad (15)$$

Proof 5: Indeed, this number also represents the number of solutions in N^m of the unknown equation $(a_1, ..., a_m)$ such that $\sum_{i=1}^{m} a_i = n$ with a_i the number of individuals in the microeconomic structure i.

We put $b_i = a_i+1$, since, $a_i \in N$, then $b_i \in N^*$, so, the previous equation becomes:

$$\sum_{i=1}^{m} b_i = n+m$$

By writing n+m as the sum of the number 1, n+m times:

$$n+m = 1+1+1 \ldots +1 \quad (n+m \text{ times})$$

Finding a solution of the equation also means choosing m-1 signs+among n+m-1 sign+in the above writing:

$$n+m = \underbrace{1+1+\ldots+1}_{b_1} \oplus \underbrace{1+1+\ldots+1}_{b_2} \oplus \ldots \oplus \underbrace{1+1+\ldots+1}_{b_m}$$

The number sought is thus, $C_{n+m-1}^{m-1} = C_{n+m-1}^{n}$.

Proposition 2: If we randomly distribute n individuals in m microeconomicstructures, then the probability that $A_{1,\ldots,d}$, A_i and $\tilde{A}_1$ are realized is:

$$P(A_{1,\ldots,d}) = \begin{cases} \dfrac{\prod_{i=1}^{d} C_{x_i+m_i-1}^{x_i}}{C_{n+m-1}^{n}} & \text{if} \sum_{i=1}^{d} x_i = n \\ 0 & \text{else} \end{cases} \quad (16)$$

$$P(A_i) = \frac{C_{n-x_i+m-m_i-1}^{n-x_i} \times C_{x_i+m_i-1}^{x_i}}{C_{n+m-1}^{n}} \quad (17)$$

$$P(\tilde{A}_i) = \sum_{k_i=0}^{x_i} \frac{C_{n-k_i+m-m_i-1}^{n-k_i} \times C_{k_i+m_i-1}^{k_i}}{C_{n+m-1}^{n}} \quad (18)$$

Proof 6: It is clear that $P(A_{1,\ldots,d}) = 0$ if $\sum_{i=1}^{d} x_i \neq n$. If not according to the previous lemma there are $C_{x_i+m_i-1}^{x_i}$ ways to distribute the x_i individuals in the m $m_i$croeconomic structures of the type t_i, so:

$$\text{Card } A_{1,\ldots,d} = \prod_{i=1}^{d} C_{x_i+m_i-1}^{x_i}$$

We use card $\Omega = C_{n+m-1}^{n}$, we get:

$$p(A_{1,\ldots,d}) = \frac{\prod_{i=1}^{d} C_{x_i+m_i-1}^{x_i}}{C_{n+m-1}^{n}}$$

For $P(A_i)$, we have $C_{x_i+m_i-1}^{x_i}$ ways to place the x_i individuals in the m_i microeconomic structures of the type t_i and $C_{n-x_i+m-m_i-1}^{n-x_i}$ ways to place the $(n-x_i)$ left individuals in the $(m-m_i)$ microeconomic structures left, so:

$$P(A_i) = \frac{C_{n-x_i+m-m_i-1}^{n-x_i} \times C_{x_i+m_i-1}^{x_i}}{C_{n+m-1}^{n}}$$

Remark 1: An arithmetic calculation shows that:

$$P(A_{1\ldots d}) < 1 \text{ and } P(A_i) < 1$$

Indeed:

$$\prod_{i=1}^{d} C_{x_i+m_i-1}^{x_i} = \frac{(x_1+m_1-1)!(x_2+m_2-1)!\ldots(x_d+m_d-1)!}{x_1!x_2!\ldots x_d!(m_1-1)!(m_2-1)!\ldots(m_d-1)!}$$

We have:

$$C_{n+m-1}^{n} = \frac{(x_1+x_2+\ldots+x_d+m_1+m_2+\ldots+m_d-1)!}{(x_1+x_2+\ldots+x_d)!(m_1+m_2+\ldots+m_d-1)!}$$

Notice that:

$$(x_1+x_2+\ldots+x_d)! = 1\times 2\times \ldots \times x_1(x_1+1)\ldots(x_1+x_2)(x_1+x_2+1)\ldots \ldots(x_1+\ldots+x_{d-1}+1)\ldots(x_1+\ldots+x_d)$$

So:

$$(x_1+x_2+\ldots+x_d)! > x_1!x_2!\ldots x_d!$$

In the same way, we show that:

$$(m_1+m_2+\ldots+m_d-1)!>(m_1-1)!(m_2-1)!\ldots(m_d-1)!$$

And:

$$(x_1+x_2+\ldots+x_d+m_1+m_2+\ldots+m_d-1)!>(x_1+m_1-1)!(x_2+m_2-1)!\ldots(x_d+m_d-1)!$$

Consequently:

$$\prod_{i=1}^{d} C_{x_i+m_i-1}^{x_i}<C_{n+m-1}^{n}$$

Which gives $P(A_{(1,\ldots,d)})<1$. The same trick makes it possible to show that $P(A_i)<1$.

Corollary 1: Recall that we have $(A_i)_{i=1,\ldots,d}$ a partition of A_m consequently:

$$\sum_{i=1}^{d} P(A_i)=1$$

We have:

$$P(A_i)=\sum_{(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_d)} P[X=(x_1,\ldots,x_d)]$$

It is evident that this summation is difficult and yet, we have:

$$P(A_i)=\sum_{(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_d)} P[X=(x_1,\ldots,x_d)]=\frac{C_{n-x_i+m-m_i-1}^{n-x_i}\times C_{x_i+m_i-1}^{x_i}}{C_{n+m-1}^{n}}$$

Distribution of n individuals numbered in m microeconomic structures: If we distribute n individuals numbered from 1 to n, then there are two distributions that differ only in individual's number. Note that the number of ways to place n individuals numbered in m microeconomic structures is m^n, since for each one there are m ways to choose the microeconomic structure that contains the individual. Therefore:

$$\text{card}\Omega=m^n \quad (19)$$

Remark 2: When the distribution of the individuals is random, we have C_{n+m-1}^{n}, now the notion of the order has increased the cardinal because:

$$m^n>C_{n+m-1}^{n}$$

Indeed:

$$C_{n+m-1}^{n}=\frac{A_{n+m-1}^{n}}{n!}=\frac{(n+m-1)(n+m-2)\ldots m}{n(n-1)\ldots 1}$$

So, it is enough to compare m with m+p/p+1 for p = 0,1, …., n-1. We also have:

$$m-\frac{m+p}{p+1}=\frac{p(m-1)}{p+1}\geq 0$$

Hence, the desired result.

Proposition 3: The probability that the events $A_{1,\ldots,d}$, A_i and $\tilde{A}_1$ are realized is given by:

$$P(A_{1,\ldots,d})=\begin{cases}\prod_{j=1}^{d} C_{n-\sum_{i=1}^{j-1}x_i}^{x_j}\left(\frac{m_j}{m}\right)^{x_j} & \text{if } \sum_{i=1}^{d}x_i=n\\ 0 & \text{else}\end{cases} \quad (20)$$

$$P(A_i)=C_n^{x_i}\left(1-\frac{m_i}{m}\right)^{n-x_i}\left(\frac{m_i}{m}\right)^{x_i} \quad (21)$$

$$P(\tilde{A}_1)=\sum_{k_i=0}^{x_i} C_n^{k_i}\left(1-\frac{m_i}{m}\right)^{n-k_i}\left(\frac{m_i}{m}\right)^{k_i} \quad (22)$$

Proof 7: We have $P(A_{1,\ldots,d})=0$ if $\sum_{i=1}^{d}x_i\neq n$ for the type t_1,we choose x_1 individuals among n, so, there are $C_n^{x_1}$ possibilities of choices and for each one there are, $m_1^{x_1}$ ways to place the x_1 individuals in the m_1 microeconomic structures.

For the type t_2 we choose x_2 individuals from n-x_1, $C_{n-x_1}^{x_2}$ possibilities and for each choice there are $m_2^{x_2}$ ways to place the x_2 individuals in the m_2 microeconomic structures.

In general for the type t_j we choose x_j individuals from $n-\sum_{i=1}^{j-1}x_i$, so, there are $C_{n-\sum_{i=1}^{j-1}x_i}^{x_j}$ possibilities of choices and for each one there are $m_j^{x_j}$ ways to place the x_j individuals in the m_j microeconomic structures. So:

$$\text{card}(A_{1,\ldots,d})=\prod_{j=1}^{d} C_{n-\sum_{i=1}^{j-1}x_i}^{x_j}(m_j)^{x_j}$$

Consequently:

$$P(A_{1,\ldots,d})=\prod_{j=1}^{d} C_{n-\sum_{i=1}^{j-1}x_i}^{x_j}\left(\frac{m_j}{m}\right)^{x_j}$$

To compute $P(A_i)$, it is enough to compute card A_i, so, to choose x_i individuals from n, there are $C_n^{x_i}$ possibilities and for each choice there are $m_i^{x_i}$ ways to place the x_i individuals in the m_{ii} microeconomic structures and there are $(m-m_i)^{n-x_i}$ ways to place n-x_i other individuals among m-m_i microeconomic structures left, this gives:

$$\text{card}\left(A_i\right) = C_n^{x_i} m_i^{x_i} (m - m_i)^{n-x_i}$$

And we get:

$$P\left(A_i\right) = \frac{C_n^{x_i} m_i^{x_i} (m - m_i)^{n-x_i}}{m^n} = C_n^{x_i} (1 - \frac{m_i}{m})^{n-x_i} \left(\frac{m_i}{m}\right)^{x_i}$$

Theorem 4: When we distribute n individuals numbered from 1 to n in m microeconomic structures, the random variable X_i follows a binomial (Forbes *et al.*, 2010; Pitman, 1997) distribution characterized by:

$$B\left(n, ß_i = \frac{m_i}{m}\right) \quad (23)$$

Indeed, if we put $\beta_i = m_i/m$ in the Eq. 21, we can rewrite the above as:

$$P(A_i) = C_n^{x_i} (1 - ß_i)^{n-x_i} (ß_i)^{x_i} \quad (24)$$

Remark 3: The term $\prod_{j=1}^{d} C^{x_j}_{n-\sum_{i=1}^{j-1} x_i}$ noted $C_n^{x_1, x_2, \ldots, x_d}$ is a multinomial term that appears in the generalization of the formula of the binome:

$$(a_1 + a_2 + \ldots + a_d)^n = \sum_{x_1 + x_2 + \ldots + x_d = n} C_n^{x_1, x_2 \ldots x_d} a_1^{x_1} a_2^{x_2} \ldots a_d^{x_d}$$

The probability $P(A_{1,\ldots,d})$ is also written as:

$$P(A_{1 \ldots d}) = \frac{C_n^{x_1, x_2 \ldots x_d} m_1^{x_1} m_2^{x_2} \ldots m_d^{x_d}}{m^n}$$

Since, $m^n = (m_1 + m_2 + \ldots m_d)^n$, we get $P(A_{1,\ldots,d}) < 1$

Distribution of n individuals in m microeconomic structures knowing that each one can contain only one individual at most: We are now working on the assumption that each microeconomic structure can contain only one individual at most. In this situation, each one contains 0 or 1 individual (necessarily with this distribution we impose that n< m), so:

$$\text{card}\Omega = A_m^n \quad (25)$$

Proposition 4: With the distribution of n individuals in m microeconomic structures knowing that everyone can contain at most one individual, we have:

$$P(A_{1,\ldots,d}) = \begin{cases} \dfrac{\prod_{j=1}^{d} C^{x_j}_{n-\sum_{i=1}^{j-1} x_i} A_{m_j}^{x_j}}{A_m^n} & \text{if } \sum_{i=1}^{d} x_i = n \\ 0 & \text{else} \end{cases} \quad (26)$$

$$P(A_i) = \frac{C_{m_i}^{x_i} C_{m-m_i}^{n-x_i}}{C_m^n} \quad (27)$$

$$P\left(A_1\right) = \sum_{k_i=0}^{x_i} \frac{C_{m_i}^{k_i} C_{m-m_i}^{n-k_i}}{C_m^n} \quad (28)$$

Proof 8: It is obvious that the event is impossible if $x_i > m_i$ or $\sum_{i=1}^{d} x_i \neq n$. For the opposite case we consider the type t_1 we choose x_1 individuals among n therefore, there are $C_n^{x_1}$ possibilities and for each choice there are $A_{m_1}^{x_1}$ ways to choose the microeconomic structures in which the x_1 will go.

For the type t_2, we choose x_2 individuals so, there are $C_{n-x_1}^{x_2}$ possibilities and for each choice there are $A_{m_2}^{x_2}$ ways to choose the microeconomic structures in which the x_2 will go.

In general for the type t_j, we choose x_j individuals, so, there are $C^{x_j}_{n-\sum_{i=1}^{j-1} x_i}$ possibilities and for each Choice there are $A_{m_j}^{x_j}$ ways to choose the microeconomic structures in which the x_j will go, so:

$$\text{Card } A_{1,\ldots,d} = \prod_{j=1}^{d} C^{x_j}_{n-\sum_{i=1}^{j-1} x_i} A_{m_j}^{x_j}$$

And consequently:

$$P(A_{1,\ldots,d}) = \begin{cases} \dfrac{\prod_{j=1}^{d} C^{x_j}_{n-\sum_{i=1}^{j-1} x_i} A_{m_j}^{x_j}}{A_m^n} & \text{if } \sum_{i=1}^{d} x_i = n \\ 0 & \text{else} \end{cases}$$

Let us calculate card A_i, for that we choose x_i individuals among n, so there are $C_n^{x_i}$ possibilities and for each choice there are $A_{m_i}^{x_i}$ ways to choose the microeconomic structures that contain them and there are $A_{m-m_i}^{n-x_i}$ ways to choose the other $m-m_i$ microeconomic structures that contain the $n-x_i$ other left individuals. Therefore: card $A_i = C_n^{x_i} A_{m_i}^{x_i} A_{m-m_i}^{n-x_i}$. So:

$$P(A_i) = \frac{C_n^{x_i} A_{m_i}^{x_i} A_{m-m_i}^{n-x_i}}{A_m^n} = \frac{C_{m_i}^{x_i} C_{m-m_i}^{n-x_i}}{C_m^n} \quad (29)$$

Theorem 5: The random variable X_i follows a hypergeometric distribution $H(n, m_i, m)$ (Walk, 2007).

Remark 4: The probability $P(A_{1,\ldots,d})$ can also be written as:

$$P(A_{1,\ldots,d}) = \frac{\prod_{j=1}^{d} C_{m_j}^{x_j}}{C_m^n} \quad (30)$$

Because:

$$\frac{\prod_{j=1}^{d} C_{n-\sum_{i=1}^{j-1} x_i}^{x_j} A_{m_j}^{x_j}}{A_m^n} = \frac{\prod_{j=1}^{d} C_{m_j}^{x_j}}{C_m^n}$$

Remark: For $P(\tilde{A}_{1,\ldots,d})$ in the three cases we have:

$$P(\tilde{A}_{1,\ldots,d}) = \sum_{\sum_{i=1}^{d} k_i = 0}^{\sum_{i=1}^{d} x_i} P[X_1 = k_1, \ldots, X_d = k_d]$$

$$= \begin{cases} P[X_1 = k_1, \ldots, X_d = k_d] & \text{if } \sum_{i=1}^{d} k_i = \sum_{i=1}^{d} x_i \\ 0 & \text{if } 0 \leq \sum_{i=1}^{d} k_i < \sum_{i=1}^{d} x_i \end{cases}$$

Application 2: A direct application of this study is choosing the poor defined by Adawo (2010) and Spicker *et al.* (2007) as the individuals and the incoming generating activities (Chikina *et al.*, 2007) for the microeconomic structures.

CONCLUSION

One of the direct applications of this research is the estimation of the guarantee fund of the state when we seek to develop projects IGA with a quadruplet operation (IGA capital, share, credit, guarantee), besides the interest of the high-order moments is well known as statistical analysis tool, the calculation of these moments is often complicated, just think of the poisson distribution. In this study, we demonstrated the link between the high-order moments and the factorial moments using various ways of calculation in particular the work that we published in Mohammed *et al.* (2016) and the formula of Faa Di Bruno. The recursive calculation provides a more simplified way to determine the distributions.

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