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Dynamic Response of a Timoshenko Shaft with a Rigid Disk

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Abstract: This study presents the method to study the free and forced dynamic response of a shaft of a Pelton turbine unit. Equation of motion for the bending vibration of the shat of the Pelton turbine assembly is developed by using Hamilton principle by assuming Pelton wheel as a rigid disk attached on a Timoshenko shaft. Impact provided by the water jet is represented by Fourier series expansion. Governing equations of the system in terms of coupled partial differential equations are converted into coupled modal equations by using assumed mode method. Then, these coupled ordinary differential equations are solved for free and forced response of the system.

Key words: Pelton turbine, Timoshenko shaft, free response, forced response, vibration, critical speed

INTRODUCTION

Most of the machines used in industries, power plants consists of rotating components. Dynamic behaviour of such systems are studied by developing a rotodynamic model which consists of mainly disk, shaft and bearings. To improve the performance and reliability of such systems dynamic phenomena occurring during its operations should be investigated. The dynamic response of such shaft-disk system depends upon parameters, interaction between system components, interaction of system with intervening medium and operating conditions.

Many researchers have attempted to investigate different aspects of such system to understand the dynamic behaviour of the system. Some researchers have studied dynamic behaviour of shaft only, some have studied the dynamic behaviour of the shaft and disk assembly while some have studied effect of different external interactions that occur on the bearing or on the disk.

Thomas and Abbas (1976) have developed a finite element model for the stability analysis of Timoshenko beam subjected to periodic axial loads and have studied the effect of the shear deformation on the static buckling loads. They have also investigated regions of dynamic instability and have presented values of critical loads for beams with various shear parameters.

Nelson (1980) has used Timoshenko beam theory for establishing the shape functions and thereby including transverse shear effects for the simulation of rotor systems. He has compared the solution obtained from the finite element method with the classical closed form Timoshenko beam theory analysis for nonrotating and rotating shafts.

Lee and Jei (1988) have applied modal analysis to continuous rotor systems with various boundary conditions which include isotropic and anisotropic natural boundary conditions. They have determined the whirl speeds and mode shapes, backward and forward of a rotating shaft for spin speeds and boundary conditions vary and have also calculated the unbalance responses by using modal analysis. The effects of asymmetry in boundary conditions on the system dynamic characteristics were also investigated.

Jei and Lee (1992) have performed modal analysis of an asymmetrical rotor-bearing system which consists of asymmetrical Rayleigh shafts, asymmetrical rigid disks and isotropic bearings. They have developed a solution method for the vibration analysis of a rotating uniform asymmetrical shaft which can determine the whirl speeds and mode shapes of the uniform asymmetrical shaft.

Choi *et al*. (1992) have derived the equations of motion of a flexible rotating shaft by introducing gyroscopic moments describing the flexural vibration in two orthogonal planes and the torsional vibration of a straight rotating shaft with dissimilar lateral principal moments of inertia and subject to a constant compressive axial load. They have also presented an approach for calculating correctly the effect of an axial load for a Timoshenko beam based on the change in length of the centroidal line.

Han and Zu (1992) have analyzed a spinning Timoshenko beam subjected to a constant moving load

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using a modal expansion technique and have determined the dynamic quantities such as natural frequencies, mode shapes and system response. Numerical simulations were also performed to demonstrate the characteristics of the response.

Han and Zu (1993) have studied the dynamics of a simply supported, spinning Timoshenko beam subjected to a moving load using a modal analysis technique. They have shown that the simply supported spinning Timoshenko beams possess two pairs of natural frequencies. Closed-form expressions for natural frequencies and the system transient response were presented using this simplified theory.

Lee (1995) has analyzed the dynamic response of a rotating shaft subject to axial force and moving loads by using Timoshenko beam theory and the assumed mode method. He has derived equations of motion in matrix form by using Hamilton's principle and has investigated the influences of the rotational speed of the shaft, the axial speed of the loads and the Rayleigh coefficient and compared with the available results.

Lee and Yun (1996) have analyzed the effect of the direction of application and magnitude of loads on the stability and natural frequency of flexible rotors subjected to non-conservative torque and force. They have derived the stability criterion from the energy and variational principle and a general Galerkin method in which admissible functions were used for numerical analysis.

Mealnson and Zu (1998) have performed vibration analysis of an internally damped rotating shaft using Timoshenko beam theory with general boundary conditions. They have derived the equations of motion including the effects of internal viscous and hysteretic damping and have obtained exact solutions for the complex natural frequencies and complex normal modes are six classical boundary conditions.

Wong and Zu (1999) have studied the dynamic behaviour of a simply-supported spinning Timoshenko shaft with coupled bending and torsion. They have performed the analysis by transforming the set of nonlinear partial differential equations of motion into a set of linear ordinary differential equations. They obtained analytical solution for the set of time varying ordinary differential equations in terms of Chebyshev series.

Fung and Hsu (2000) have formulated governing equations for the rotating flexible-Timoshenko-shaft/flexible-disk coupling system by introducing the kinetic and strain energies and the virtual research done by the Eddy-current brake system using Hamilton's principle. They have found that the Eddy-current brake system can be used to decrease speed and suppress flexible and shear vibrations simultaneously.

Huai-Liang (2002) has presented the dynamic simulation for an axial moving flexible rotating shafts which have large rigid motions and small elastic deformation. He has derived the equations of motion and associated boundary conditions using Hamilton principle and has obtained the solution is obtained by using the perturbation approach and assuming mode method. He has also investigated the influence of the axial rigid motion, shear deformation, slenderness ratio and rotating speed on the dynamic behaviour of Timoshenko rotating shaft.

Sinha (2005) has studied the dynamic response of a Timoshenko beam under repeated pulse loading. He has derived the basic dynamical equations for a rotating radial cantilever Timoshenko beam clamped at the hub in a centrifugal force field. Rayleigh-Ritz method with a set of sinusoidal displacement shape functions has been used to determine stiffness, mass and gyroscopic matrices of the system analytically. Transient response of the beam with the tip deforming due to rub has been discussed in terms of the frequency shift and non-linear dynamic response of the rotating beam.

Salarieh and Ghorashi (2006) have analyzed the free vibration of a cantilever Timoshenko beam with a rigid tip mass. They have shown that the beam can be exposed to both torsional and planar elastic bending deformations. They have solved the governing equation numerically to study the dependency of natural frequencies on various parameters of the tip.

Ozsahin *et al*. (2014) have presented an analytical modelling and an analysis approach for asymmetric multi-segment rotor bearing systems. They have obtained sub-segment Frequency Response Functions (FRFs) analytically and sub-segment FRFs obtained have been coupled by using receptance coupling method. They have shown that using analytical model and receptance coupling, compared with FEM, reduces computational time drastically without losing accuracy.

Bhaskar and Saheb (2015) have solved the problem of large amplitude free vibrations of a uniform shear flexible hinged beam at higher modes with ends immovable to move axially. They have compared numerical results obtained with those obtained through finite element and other continuum methods for the fundamental mode and have found close to each other.

Mirtalaie and Hajabasi (2016) have performed the linear lateral free vibration analysis of the rotor is performed based on the Timoshenko beam theory including the effects of rotary inertia, gyroscopic effects and shear deformations. They have shown that the system can be classified as the parametrically excited systems because of the mutual interaction of shear and Euler angles which leads to some variable coefficient terms appeared in the kinetic energy of the system. They have investigated the free vibration behaviour of parametrically excited system by perturbation method and compared with the common Rayleigh, Timoshenko and higher-order shear deformable spinning beam models in the rotordynamics.

Most of the earlier papers have focused the dynamic response of a flexible shaft with reference to inherent system parameters such as nonhomogeneity, asymmetry, inextensionability, etc. or intervention from the surroundings such as rub impact or effects of bearing properties. This study focus mainly on the dynamic response of the shaft of a system consisting of a rigid disk on a flexible shaft which can used to study the behaviour of a Pelton turbine and its assembly.

MATERIALS AND METHODS

Problem formulation: Consider a rigid disk attached to a flexible shaft with a constant spin speed of Ω as shown in Fig. 1. The axes x-z are chosen such that x is along longitudinal direction of the shaft, y is along transverse direction of shaft on the horizontal plane and z is along the transverse direction of the shaft on the vertical plane. Similarly, transverse displacements of any point of the shaft along horizontal and vertical directions are respectively v(x, t) and w(x, t). For the horizontal shaft Pelton turbine water jet acts along the y direction. Similarly, rotations of any point of the shaft about y and z axis are respectively $\phi(x, t)$ and $\psi(x, t)$. Kinetic energy of the Timoshenko shaft is given by:

$$T_s = \frac{1}{2}\rho A\int_0^L \dot{v}^2 dx + \frac{1}{2}\rho A\int_0^L \dot{w}^2 dx + \frac{1}{2}\rho A\Omega^2\int_0^L v^2 dx + \frac{1}{2}\rho A\Omega^2\int_0^L w^2 dx + \rho A\Omega\int_0^L \dot{w}v dx - \rho A\Omega\int_0^L \dot{v}w dx + \frac{1}{2}\rho I_s\int_0^L \dot{\phi} dx + \frac{1}{2}\rho I_s\int_0^L \dot{\psi}^2 dx - \frac{1}{2}\rho I_s\Omega^2\int_0^L \phi^2 dx - \frac{1}{2}\rho I_s\Omega^2\int_0^L \psi^2 dx \tag{1}$$

Similarly, kinetic energy of the rigid disk can be expressed as:

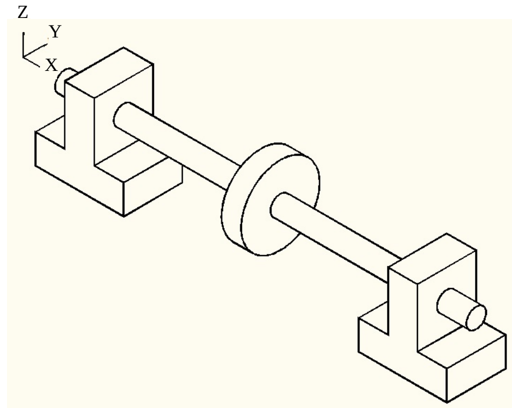

Fig. 1: Shaft-disk assembly

$$T_d = \frac{1}{2}M_d\left(\dot{v}^2\right)\Big|_{x=\frac{L}{2}} + \frac{1}{2}M_d\left(\dot{w}^2\right)\Big|_{x=\frac{L}{2}} + \frac{1}{2}M_d\Omega^2\left(v^2\right)\Big|_{x=\frac{L}{2}} + \frac{1}{2}M_d\Omega^2\left(w^2\right)\Big|_{x=\frac{L}{2}} + M_d\Omega\left(\dot{w}v\right)\Big|_{x=\frac{L}{2}} - M_d\Omega\left(\dot{v}w\right)\Big|_{x=\frac{L}{2}} + \frac{1}{2}\rho_d h I_d\left(\dot{\phi}^2\right)\Big|_{x=\frac{L}{2}} + \frac{1}{2}\rho_d h I_d\left(\dot{\psi}^2\right)\Big|_{x=\frac{L}{2}} - \frac{1}{2}\rho_d h I_d\Omega^2\left(\phi^2\right)\Big|_{x=\frac{L}{2}} - \frac{1}{2}\rho_d h I_d\Omega^2\left(\psi^2\right)\Big|_{x=\frac{L}{2}} \tag{2}$$

The strain energy of the shaft due to bending and shear deformation is then given by:

$$U_s = \frac{1}{2}EI_s\int_0^L \phi dx + \frac{1'}{2}EI_s\int_0^L (\psi')^2 dx + \frac{1}{2}kAG\int_0^L (v')^2 dx + \frac{1}{2}kAG\int_0^L (w')^2 dx + \frac{1}{2}kAG\int_0^L \phi^2 dx + \frac{1}{2}kAG\int_0^L \psi^2 dx - kAG\int_0^1 v'\phi\, dx - kAG\int_0^L w'\psi dx \tag{3}$$

Work done by the impact of jet is given by:

$$W_{ext} = F(t)(v)\Big|_{x=\frac{L}{2}} \tag{4}$$

Then, Lagrangian is given by:

$$L = T-U \tag{5}$$

Then, applying Hamilton's principle, equations of motion and associated boundary conditions are obtained as:

$$\rho A\dot{v} - \rho A\Omega^2 v - 2\rho A\Omega\dot{w} + M_d\delta\left(x-\frac{L}{2}\right)\dot{v} - M_d\Omega^2\delta\left(x-\frac{L}{2}\right)v - 2M_d\Omega\delta\left(x-\frac{L}{2}\right)\dot{w} - kAGv'' + kAG\phi' - F(t)\delta\left(x-\frac{L}{2}\right) = 0 \tag{6}$$

$$\left[\left\{kAGv'-kAG\phi\right\}\delta v\right]\Big|_0^L = 0 \tag{7}$$

$$\rho A\dot{w}-\rho A\Omega^2 w+2\rho A\Omega\dot{v}+M_d\delta\left(x-\frac{L}{2}\right)\ddot{w}- M_d\Omega^2\delta\left(x-\frac{L}{2}\right)w+2M_d\Omega\delta\left(x-\frac{L}{2}\right)\dot{v}- kAGw''+kAG\psi' = 0 \tag{8}$$

$$\left[\left\{kAGw'-kAG\psi\right\}\delta v\right]\Big|_0^L = 0 \tag{9}$$

$$\rho I_s\ddot{\phi}+\rho I_s\Omega^2\phi+\rho_d hI_d\delta\left(x-\frac{L}{2}\right)\ddot{\phi}+\rho_d hI_d\Omega^2\delta\left(x-\frac{L}{2}\right)\phi- EI_s\phi''+kAG\phi-kAGv' = 0 \tag{10}$$

$$\left[EI_s\phi'\delta\phi\right]\Big|_0^L = 0 \tag{11}$$

$$\rho I_s\ddot{\psi}+\rho I_s\Omega^2\psi+\rho_d hI_d\delta\left(x-\frac{L}{2}\right)\ddot{\psi}+\rho_d hI_d\Omega^2\delta \left(x-\frac{L}{2}\right)\psi-EI_s\psi''+kAG\psi-kAGw' = 0 \tag{12}$$

$$\left[EI_s\psi'\delta\psi\right]\Big|_0^L = 0 \tag{13}$$

Using boundary conditions for the simply supported shaft which obviously satisfy the boundary conditions given by Eq. 7, 9, 11 and 13:

$$v(x, t) = \sum V_n(t)\sin\left(\frac{n\pi x}{L}\right) \tag{14}$$

$$w(x,t) = \sum W_n(t)\left(\frac{n\pi x}{L}\right) \tag{15}$$

$$\phi(x, t) = \sum \Phi_n(t)\cos\left(\frac{n\pi x}{L}\right) \tag{16}$$

$$\psi(x,t) = \sum \psi_n(t)\cos\left(\frac{n\pi x}{L}\right) \tag{17}$$

Using orthogonality conditions, equations of motion for each mode can be expressed as:

$$M_n\ddot{V}_n(t)-C_n\dot{W}_n(t)+K_nV_n(t)-P_n\Phi_n(t)-F_n(t) = 0 \tag{18}$$

$$M_n\ddot{W}_n(t)+C_n\dot{V}_n(t)+K_nW_n(t)-P_n\psi_n(t) = 0 \tag{19}$$

$$I_n\ddot{\Phi}_n(t)+Kt_n\Phi_n(t)-P_nV_n(t) = 0 \tag{20}$$

$$I_n\ddot{\psi}_n(t)+Kt_n\psi_n(t)-P_nW_n(t) = 0 \tag{21}$$

Where:

$$M_n = \frac{1}{2}\rho AL+M_d\sin^2\left(\frac{n\pi}{2}\right) \tag{22}$$

$$C_n = \rho AL\Omega+2M_d\Omega\sin^2\left(\frac{n\pi}{2}\right) \tag{23}$$

$$K_n = -\frac{1}{2}\rho AL\Omega^2-M_d\Omega^2\sin^2\left(\frac{n\pi}{2}\right)+\frac{1}{2L}kAGn^2\pi^2 \tag{24}$$

$$P_n = \frac{1}{2}kAG\,n\pi \tag{25}$$

$$F_n = F_j(t)\sin\left(\frac{n\pi}{2}\right) \tag{26}$$

$$I_n = \frac{1}{2}\rho I_sL+\rho_d hI_d\cos^2\left(\frac{n\pi}{2}\right) \tag{27}$$

$$Kt_n = \frac{1}{2}\rho I_sL\Omega^2+\rho_d hI_d\Omega^2\cos^2\left(\frac{n\pi}{2}\right)+ \frac{1}{2}\frac{EI_sn^2\pi^2}{L}+\frac{1}{2}kAGL \tag{28}$$

Rearranging Eq. 20 and 21 for $\Phi_n(t)$ and $\Psi_n(t)$, respectively:

$$\Phi_n(t) = \frac{M_n}{P_n}\ddot{V}_n(t)-\frac{C_n}{P_n}\dot{W}_n(t)+\frac{K_n}{P_n}V_n(t)-\frac{1}{P_n}F_n(t) \tag{29}$$

$$\Psi_n(t) = \frac{M_n}{P_n}\ddot{W}_n(t)+\frac{C_n}{P_n}\dot{V}_n(t)+\frac{K_n}{P_n}W_n(t) \tag{30}$$

Substituting $\Phi_n(t)$ and $\Psi_n(t)$, respectively into Eq. 18 and 19:

$$I_n\frac{M_n}{P_n}\frac{d^4V_n(t)}{dt^4}-I_n\frac{C_n}{P_n}\frac{d^3W_n(t)}{dt^3}+\left\{I_n\frac{K_n}{P_n}+Kt_n\frac{M_n}{P_n}\right\} \frac{d^2V_n(t)}{dt^2}-Kt_n\frac{C_n}{P_n}\frac{dW_n(t)}{dt}+\left\{Kt_n\frac{K_n}{P_n}-P_n\right\}V_n(t)- \frac{Kt_n}{P_n}F_n(t)-\frac{I_n}{P_n}\frac{d^2}{dt^2}\left\{F_n(t)\right\} = 0 \tag{31}$$

$$I_n \frac{M_n}{P_n}\frac{d^4W_n(t)}{dt^4} + I_n \frac{C_n}{P_n}\frac{d^3V_n(t)}{dt^3} + \left\{I_n \frac{K_n}{P_n} + Kt_n \frac{M_n}{P_n}\right\} \frac{d^2W_n(t)}{dt^2} + Kt_n \frac{C_n}{P_n}\frac{dV_n(t)}{dt} + \left\{Kt_n \frac{K_n}{P_n} - P_n\right\} W_n(t) = 0 \quad (32)$$

Equation 31 and 32 can also be expressed in simpler forms as:

$$\frac{d^4V_n(t)}{dt^4} - \alpha_{1n}\frac{d^3W_n(t)}{dt^3} + \alpha_{2n}\frac{d^2V_n(t)}{dt^2} - \alpha_{3n}\frac{dW_n(t)}{dt} + \alpha_{4n}V_n(t) - \alpha_{5n}F_n(t) - \alpha_{6n}\frac{d^2}{dt^2}\{F_n(t)\} = 0 \quad (33)$$

$$\frac{d^4W_n(t)}{dt^4} + \alpha_{1n}\frac{d^3V_n(t)}{dt^3} + \alpha_{2n}\frac{d^2W_n(t)}{dt^2} + \alpha_{3n}\frac{dV_n(t)}{dt} + \alpha_{4n}W_n(t) = 0 \quad (34)$$

Where:

$$\alpha_{1n} = \frac{C_n}{M_n} \quad (35)$$

$$\alpha_{2n} = \frac{K_n}{M_n} + \frac{Kt_n}{I_n} \quad (36)$$

$$\alpha_{3n} = \frac{Kt_n}{I_n}\frac{C_n}{M_n} \quad (37)$$

$$\alpha_{4n} = \frac{Kt_n}{I_n}\frac{K_n}{M_n} - \frac{P_n^2}{I_nM_n} \quad (38)$$

$$\alpha_{5n} = \frac{Kt_n}{I_nM_n} \quad (39)$$

$$\alpha_{6n} = \frac{1}{M_n} \quad (40)$$

Solution for free response: For free vibration analysis, substituting F(t) = 0, Eq. 33 and 34 reduce to:

$$\frac{d^4V_n(t)}{dt^4} - \alpha_{1n}\frac{d^3W_n(t)}{dt^3} + \alpha_{2n}\frac{d^2V_n(t)}{dt^2} - \alpha_{3n}\frac{dW_n(t)}{dt} + \alpha_{4n}V_n(t) = 0 \quad (41)$$

$$\frac{d^4W_n(t)}{dt^4} + \alpha_{1n}\frac{d^3V_n(t)}{dt^3} + \alpha_{2n}\frac{d^2W_n(t)}{dt^2} + \alpha_{3n}\frac{dV_n(t)}{dt} + \alpha_{4n}W_n(t) = 0 \quad (42)$$

Substituting:

$$V_n(t) = \bar{V}_n e^{s_n t} \quad (43)$$

and:

$$W_n(t) = \bar{W}_n e^{s_n t} \quad (44)$$

into Eq. 41 and 42, the characteristics equation of the system can be obtained in terms of system parameters as:

$$s_n^8 + (\alpha_{1n}^2 + 2\alpha_{2n})s_n^6 + (2\alpha_{1n}\alpha_{3n} + \alpha_{2n}^2 + 2\alpha_{4n})s_n^4 + (2\alpha_{2n}\alpha_4 + \alpha_{3n}^2)s_n^2 + \alpha_{4n}^2 = 0 \quad (45)$$

Equation 45 can also be expressed as a fourth order polynomial equation by substituting $s_n^2 = \lambda_n$:

$$\lambda_n^4 + (\alpha_{1n}^2 + 2\alpha_{2n})\lambda_n^3 + (2\alpha_{1n}\alpha_{3n} + \alpha_{2n}^2 + 2\alpha_{4n})\lambda_n^2 + (2\alpha_{2n}\alpha_4 + a_{3n}^2)\lambda_n + \alpha_{4n}^2 = 0 \quad (46)$$

Equation 46 gives four roots of λ_{rp} among which $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ give natural frequencies for forward and backward whirl for the mode n and $\sqrt{\lambda_3}$ and $\sqrt{\lambda_4}$ have very high values and are quite far from the operating speeds.

Solution for forced response: In this study, forced response of the Pelton turbine unit due to impact of water jet is studied. Tangential force provided by the water jet on the runner wheel can be approximated a series of periodically appearing pulses as shown in Fig. 2 where, t_2-t_1 (= t_4-t_3 = t_6-t_5 = ,...,) is the duration of each pulse which is proportional to the bucket thickness to the circumference of the equivalent runner wheel and T is the period of one revolution of the runner wheel.

Since, the force exerted by the water jet is periodic but non-harmonic, it can be converted into harmonic terms by using Fourier series expansion (Kreyszig, 2011) as:

$$F(t) = \alpha_0 + \sum_{n=1}^{\infty}\left[\alpha_n \cos\left(\frac{2n\pi}{\tau}t\right) + b_n \sin\left(\frac{2n\pi}{\tau}t\right)\right] \quad (47)$$

Where:

$$\alpha_0 = \frac{2}{\tau}\int_0^{\tau} F(t)dt \quad (48)$$

$$\alpha_n = \frac{1}{\tau}\int_0^{\tau} F(t)\cos\left(\frac{2n\pi}{\tau}t\right)dt \quad (49)$$

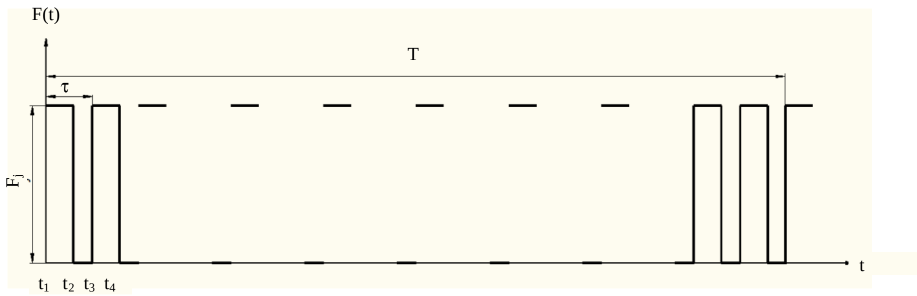

Fig. 2: Pulses of force due to water jet on Pelton turbine

$$b_n = \frac{1}{\tau}\int_0^{\tau} F(t)\sin\left(\frac{2n\pi}{\tau}t\right)dt \tag{50}$$

Substituting F(t) from Eq. 47 into Eq. 33, steady state response of the system can be determined.

RESULTS AND DISCUSSION

To present meaningful interpretations of the result numerical analysis is carried out by using the following values of system parameters (Table 1).

Numerical results for free response: By substituting all values of parameters into Eq. 46, the natural frequencies corresponding to backward whirl and forward whirl are determined. This can be presented in the form of Campbell diagram as shown in Fig. 3.

Natural frequencies of each mode corresponding to zero spin speed are the natural frequencies of first three modes of the simply supported Timoshenko beam. As the spin speed increases critical speed for backward whirl of each mode decreases whereas the critical speed for forward whirl for each mode increases. At lower speeds bending stiffness will have higher value than the stiffness due to centrifugal effect (centrifugal stiffening). During backward whirl centrifugal stiffening will act opposite to elastic restoring force and therefore critical speed for backward whirl decreases with the increase in spin speed as shown in Fig. 3. During forward whirl centrifugal stiffening will act in the same direction to elastic restoring force and therefore critical speed for forward whirl increases with the increase in spin speed as shown in Fig. 3.

The difference between the critical speeds for the forward whirl and backward whirl for a given operating speed is higher for the first and third modes of vibration whereas less for the second mode of vibration.

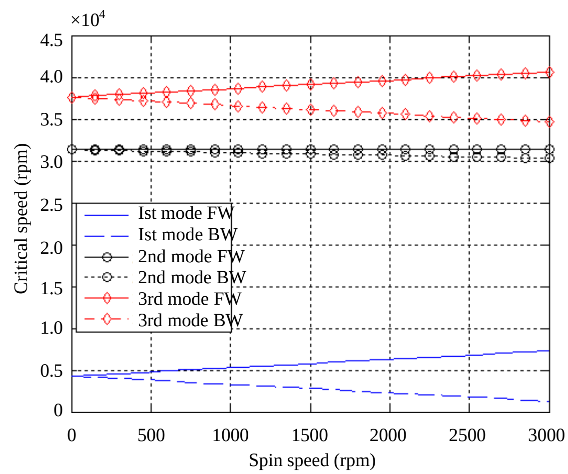

Fig. 3: Campbell diagrams for first three modes

Table 1: Parameters of the system

Parameters	Values
Density of shaft material (ρ)	7860 (kg/m^3)
Cross-sectional Area of the shaft (A)	0.8042×10^{-3} (m^2)
Length of the shaft (L)	0.52 (m)
Modulus of Elasticity of shaft material (E)	202×10^{9} (GPa)
Area moment of Inertia of the shaft section (I_s)	5.1472×10^{-8} (m^4)
Polar moment of area of the shaft section (J_{ps})	1.0294×10^{-7} (m^4)
Density of runner material (ρ_d)	8550 (kg/m^3)
Mass of runner wheel (M_d)	10.564 (kg)
Thickness of runner (h)	35 (mm)
Area moment of Inertia of the disk (I_d)	0.5527×10^{-4} (m^4)
Polar moment of area of the shaft section (J_{pd})	0.11053×10^{-3} (m^4)
Shear correction factor (k)	0.9

Numerical results for forced response: Steady state response of the system for transverse vibrations in vertical and horizontal directions are determined by considering the effects of first three modes and up to the fifth harmonics of the Fourier series representation of the jet force. The obtained responses v(x, t) and w(x, t) are presented in graphical form as.

Substituting x = L/4, into the responses obtained, the steady state response for the transverse vibration of the shaft at its quarter length is determine and which can be presented in the form response plots as shown in Fig. 4 and 5.

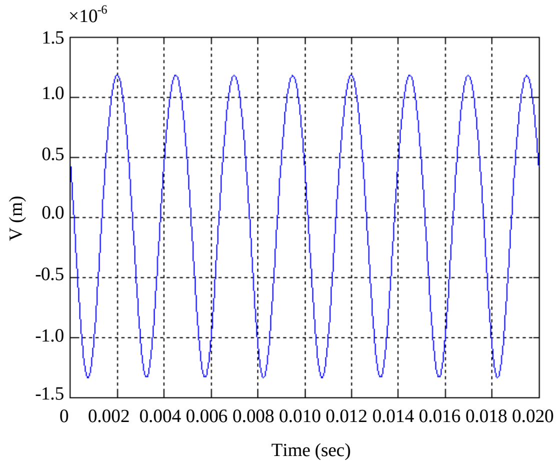

Fig. 4: Transverse displacement of shaft at x = L/4 in horizontal direction for Ω = 1500 rpm

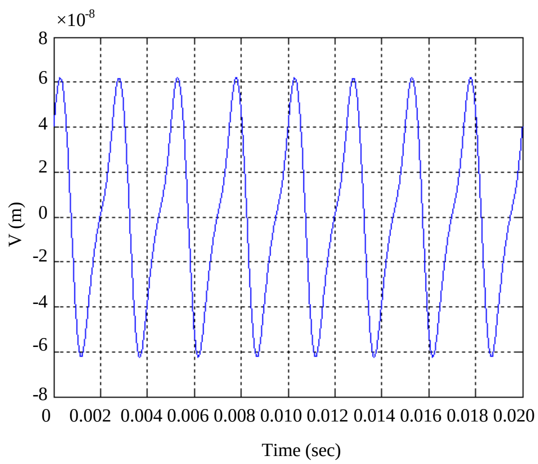

Fig. 5: Transverse displacement of shaft at x = L/4 in vertical direction for Ω = 1500 rpm

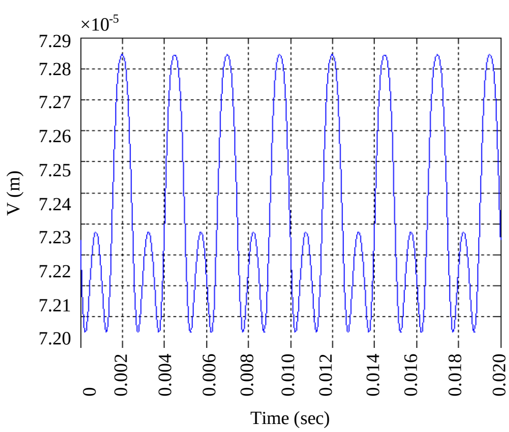

Fig. 6: Transverse displacement of disk in horizontal direction for Ω = 1500 rpm

Similarly, substituting x = L/2, the steady state response for the transverse vibration of the shaft at the mid length (at which the disk is attached) is determined and which can be presented in the form response plots as shown in Fig. 6 and 7.

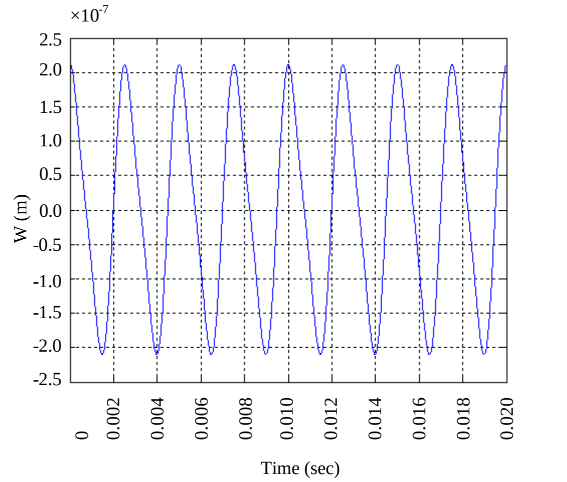

Fig. 7: Transverse displacement of disk in vertical direction for Ω = 1500 rpm

Form Fig. 4-7, it is found that the vibration amplitudes in y direction is significantly higher than that in the z direction. Higher vibration amplitude in y direction is due to the impact of jet. The vibration response in z direction is almost sinusoidal throughout the length of the shaft. The vibration response in y direction is almost sinusoidal in the region far from the disk and it has distorted form in the region near the midspan of the shaft or disk location.

CONCLUSION

In this study, dynamic behaviour of the Pelton turbine is studied by modelling it as a rigid disk attached on a Timoshenko shaft. The governing equation of the system for bending vibrations in two transverse directions are found to be coupled system of differential equations. Performing free vibration analysis, the critical speeds of the system for an operating speed of Ω = 1500 rpm for the first three modes are found to be 2787 rpm, 30821 and 36129 rpm for the backward whirl and 5786, 31746 and 39124 rpm for the forward whirl, respectively.

For the forced vibration analysis, the force provided by the water jet is approximated as s Fourier series up to the fifth harmonic components. Then steady state response for bending vibration of the system is determined by applying superposition principle. The peak amplitude of bending vibration at the midspan of the shaft (disk location) in the direction of jet for a operating speed of Ω = 1500 rpm is found to be 73 μm. Similarly, the peak amplitude of bending vibration at the midspan of the shaft (disk location) in the vertical direction for an operating speed of Ω = 1500 rpm is found to be 0.21 μm.

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