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An Information-Theoretic Approach for Multivariate Skew Laplace Normal Distributions

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ABSTRACT
Due to its flexibility, the skewness distributions (univariate and multivariate) have received widespread attention over the last two decades because their become widely used in the modelling and analysis of skewed datasets. The main goal of this paper is to introduce asymptotic expressions for entropy of multivariate skew Laplace normal distribution to deal with the issue by providing a flexible model for modeling skewness and heavy tiredness simultaneously. Thus, we extend this study to the class of mixture model of these distributions. In addition, upper and lower bounds of entropy is determined for proposed models. Finally, we give a real data examples to illustrate the behavior of information. A simulation study and a real data example are also provided to illustrate the information behavior of MSLN and MMSLN distributions for modeling data sets in multivariate settings.

INTRODUCTION

The mixture models are an important statistical tool for many applications such as data mining, density estimation, medicine, pattern recognition, image processing and satellite imaging etc[1-3].

In 1996 Azzalini[4] proposed the multivariate skew normal distribution as an alternative to multivariate normal distribution to deal with skew in the data. After a short period of time, Genton and Loperfido[5] introduced the generalization of multivariate skew normal distribution as follows:

$$f_d(y; \theta, S) = 2\varphi(y; \theta, S)\times\Psi(y-\theta),\ y\in R^d \qquad (1)$$

where, $\varphi(y; \theta, S)$ is a probability density function of multivariate normal distribution and $0\leq\Psi(y)\leq 1$ such that $\Psi(-y)=1-\Psi(y)$, for any $y\in R^d$. Clearly, if we take $\Psi(y)=1/2$ then Y has a multivariate normal distribution. As a special case if we take Ψ is a distribution function such as normal, Cauchy, Laplace, Logistic or other distributions such that $\Psi(y)=G(\delta' y)$ then we get on the generalized skew normal distribution. Huang and Gupta in derived the explicit forms of moment generated function of multivariate generalized skew distributions. Some the main properties of multivariate skew normal Cauchy distribution $MSNC_d$ are discussed by Kahrari *et al*.[6]. Lin *et al*.[7] proposed the development of mixture of skew normal and skew t-models. Mixture models of multivariate skew normal and skew t-distributions were studied by Pyne *et al*.[8]. Lee[1] provided an overview of developments of a mixture of skew t-distributions.

On the other hand, Shanon[9] presented measure to quantify the uncertainty of an event. Rényi[10] generalized this measure for probability distribution which means the sensitive to the fine details of a density function. Contreras-Reyes and Cortés[2] discussed the Rényi entropy of flexible class of skew normal distributions. Wood introduced some results for the Rényi entropy of the totally asymmetric exclusion process. Also, they calculated explicitly Rényi entropy whereby the squares of configuration probabilities are summed. It is important to refer that Contreras-Reyes and Cortés[2] showed the bounds and approximation of Rényi entropy of a class of mixture models of multivariate skew Gaussian by using the multinomial theorem and generalized HÖlder's inequality. In fact, there are no analytical expressions for the Rènyi entropy of the mixture model, therefor we consider its bounds exist. Similarity in the case of fractional relative entropy an analytic evolution of Rènyi entropy is also impossible. More recently, Abid and Quaez *et al*.[11] studied in detail the measure of information (entropy) in cases multivariate skew Laplace distributions and they extended our study to the class of mixture model of multivariate skew Laplace distributions. Also, the bounds of entropy for mixture model were determined.

Mixture models of multivariate skew normal Cauchy distributions were prosed by Abid and Quaez[12]. In addition, they derived the explicit expression of Rényi entropy of these models. The structure of this paper was designed according to the following fundamental aims:

- We determine and investigate the exact expression for Shannon and Rènyi entropies of the skew Laplace normal distributions in multivariate analysis
- We drive and identify the best approximate expression for Shannon and Rènyi entropies of mixture model of distributions by using generalized HÖlder's inequality and some properties of multinomial theorem.
- We find the bounds of entropy for mixture model of distributions
- We give and discuss a real data examples to illustrate the behavior of entropy of the mixture models under consideration

The remainder of this study is organized as follows: In section 2. we begin with a preliminary material of mixture models and the measures of information. Section 3. provides a description of asymptotic expression for entropy of multivariate skew normal Cauchy distributions by using some methods of numerical integration such as Monte Carlo and importance sampling methods. By using some theorems which related to multinomial theorem and generalized HÖlder's inequality, in section 4. we find upper and lower bounds and also we study an approximate Rényi entropy for a mixture of multivariate skew normal Cauchy distributions. The conclusion of this paper is shown in Section 5.

Preliminary material: In this section, we introduce some basic definitions, notations and lemmas related to entropy theory and mixture models of distributions.

Definition 1: Let $X\in R^d$ be a continuous random vector with probability density function $f(x, \theta)$. Then, the Shannon entropy of X is given by:

$$H(X; \theta) = -E(\ln(f(x; \theta))) \qquad (2)$$

Definition 2: Let $X\in R^d$ be a continuous random vector with probability density function $f(x; \theta)$. Then, A Rényi entropy with αth-order of X is defined as:

$$R_\alpha(X;\theta) = \begin{Bmatrix} \frac{1}{1-\alpha}\ln\left(E\left(f(x;\theta)\right)^{\alpha-1}\right), 0<\alpha<\infty, \alpha\neq 1 \\ -E\left(\ln\left(f(x;\theta)\right)\right) \qquad \alpha = 1 \end{Bmatrix} \qquad (3)$$

The relationship between the Definitions in 1 and 2 is obtained by the limit $H(X;\theta) = \lim_{\alpha\to 1} R_\alpha(X;\theta)$. Translation does not change the entropy H(X+C) = H(X)+C where, C is constant.

Proposition 1:. Suppose that $0<\alpha_1<\alpha_2<\infty$, then $R_{\alpha 1}(X;\theta)\geq R_{\alpha 2}(X;\theta)$ and the equality is hold if and only if the distribution of X is uniformly.

Proof: Firstly, Assume that $\alpha \neq 1$, then the derivative of $R_\alpha(X;\theta)$ with respect to α is given as:

$$\frac{\partial}{\partial\alpha}R_\alpha(X;\theta)=\frac{E\left(\left(f(x;\theta)\right)^{\alpha-1}\ln\left(f(x;\theta)\right)\right)}{(1-\alpha)E\left(f(x;\theta)\right)^{\alpha-1}}+\frac{\ln\left(E\left(f(x;\theta)\right)^{\alpha-1}\right)}{(1-\alpha)^2}$$

The second part in the right side can be written as:

$$\frac{\ln\left(E\left(f(x;\theta)\right)^{\alpha-1}\right)}{(1-\alpha)^2}=\frac{E\left(\left(f(x;\theta)\right)^{\alpha-1}\ln\left(E\left(f(x;\theta)\right)^{\alpha-1}\right)\right)}{(1-\alpha)^2\,E\left(f(x;\theta)\right)^{\alpha-1}}$$

Therefore:

$$\frac{\partial}{\partial\alpha}R_\alpha(X;\theta)=\frac{1}{(1-\alpha)^2}\left\{\frac{E\left(\left(f(x;\theta)\right)^{\alpha-1}\ln\left(f(x;\theta)\right)^{\alpha-1}\right)}{E\left(f(x;\theta)\right)^{\alpha-1}}+\frac{E\left(\left(f(x;\theta)\right)^{\alpha-1}\ln\left(E\left(f(x;\theta)\right)^{\alpha-1}\right)\right)}{E\left(f(x;\theta)\right)^{\alpha-1}}\right\}$$

$$=\frac{-1}{(1-\alpha)^2}\left\{\frac{E\left(\left(f(x;\theta)\right)^{\alpha-1}\ln\left(\frac{\left(f(x;\theta)\right)^{\alpha-1}}{E\left(f(x;\theta)\right)^{\alpha-1}}\right)\right)}{E\left(f(x;\theta)\right)^{\alpha-1}}\right\}$$

Define the probability density function g as:

$$g(x;\theta)=\frac{\left(f(x;\theta)\right)^{\alpha}}{E\left(f(x;\theta)\right)^{\alpha-1}}$$

Then:

$$\frac{\partial}{\partial\alpha}R_\alpha(X;\theta)=\frac{-1}{(1-\alpha)^2}\left\{E_g\left(\ln\left(\frac{g(x;\theta)}{f(x;\theta)}\right)\right)\right\}$$

But:

$$E_g\left(\ln\left(\frac{g(x;\theta)}{f(x;\theta)}\right)\right)$$

Not negative value therefor, either:

$$E_g\left(\ln\left(\frac{g(x;\theta)}{f(x;\theta)}\right)\right)>0$$

Then: R_α is a decreasing with respect to α or:

$$E_g\left(\ln\left(\frac{g(x;\theta)}{f(x;\theta)}\right)\right)=0$$

this implies that f = g almost everywhere, hence by assumption $\alpha \neq 1$ then f is only uniformly distributed. Conversely, if f is uniform distribution then it can be written as:

$$f(x;\theta)=\begin{cases}\frac{1}{Vol(B)}, & x\in B\\ 0, & x\notin B\end{cases}$$

For some measurable subset B in R^n. then:

$$R_\alpha(X;\theta)=\frac{1}{1-\alpha}\ln\left(Vol(B)^{\alpha-1}\right)=\ln\left(Vol(B)\right)$$

is not depend on the value of α.

The following proposition shows that for any location, scale model the entropy does not depend upon location parameter.

Proposition 2: Let $f_x(x;\theta,S)$ be a location scale probability density function with location vector $\theta\in R^d$, $S\in R^{d\times d}$ is dispersion matrix and let:

$$X_0=S^{-\frac{1}{2}}(X-\theta)$$

Be a standardized version of X. Then the Rènyi entropy does not depend on θ.

Proof: The probability density function f can be written as:

$$f_x(x;\theta,S)=\left(\det(S)\right)^{-\frac{1}{2}}f_{x_0}\left(S^{-\frac{1}{2}}(X-\theta),0,I\right)$$

Therefore, the Rènyi entropy of X appears as:

$$R_\alpha(X;\theta,S)=\ln\left(\det(S)\right)^{\frac{1}{2}}+\frac{1}{1-\alpha}\ln\left(E\left(f(x_0;0,I)\right)\right)^{\alpha-1}$$

$$=\ln\left(\det(S)\right)^{\frac{1}{2}}+R_\alpha(X_0;0,I)$$

Proposition 3: Let $X_0\sim MN_d(\theta, S)$. Then the Rényi entropy of X_0 can be written in the following form:

$$R_\alpha(X_0;\theta,S)=\begin{cases}\frac{1}{2}\ln\left(\det\left(2\pi\exp(1)S\right)\right), & \alpha=1\\ \frac{1}{2}\ln\left(\det(2\pi S)\right)-\frac{d}{2(1-\alpha)}\ln(\alpha), & 0<\alpha<\infty,\alpha\neq 1\end{cases}\quad(4)$$

Lemma 1: If a random vector $X\in R^d$ (not necessary normal) has zero mean and covariance matrix $\mathcal{M}$ = E(XX'), then the following inequality is accomplished[13]:

$$H(X;\theta) \le \frac{1}{2}\ln\left(\det\left(2\pi\exp(1)\mathcal{M}\right)\right) \quad (5)$$

With equality if and only if $X \sim MN_d(0, S)$.

Entropy of multivariate skew laplace normal distributions: This section includes the complete derivation of simple expressions for entropy of multivariate skew Laplace normal distribution. Some properties of transformations and integrations are used. Also, we give an illustrative example explains the relationship between the parameter αth order of entropy and skewness parameter δ with the values of R?nyi entropy. We would also like to refer that, we use some methods of numerical integrations such as Monte Carlo and importance sampling methods to find the expression of entropy.

Multivariate skew Laplace distribution has been proposed by Azzalini and Capitaino[14]. A random vector $X \in R^d$ has a multivariate skew Laplace distribution with location vector $\theta \in R^d$, scale matrix $S \in R^{d\times d}$ and skewness vector $\delta \in R^d$, denoted by ($X \sim MSLN_d(\theta, S, \delta)$), if its density is given by:

$$f_d(x;\theta,S,\delta) = \frac{\det(S)^{-\frac{1}{2}}}{2^d\beta(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\exp\left\{\begin{matrix}-\sqrt{(1+\delta' S^{-1}\delta)(x-\theta)' S^{-1}(x-\theta)}\\ +(x-\theta)' S^{-1}\delta\end{matrix}\right\} \quad (6)$$

We say that a random variable $Y \in R$ has the skew generalized Laplace distribution denoted by $Y \sim SGL(\theta, S, \delta_1, \delta_2)$ if its density function is in the following the form:

$$k(y;\theta,S,\delta_1,\delta_2) = 2g(y;\theta,S)\times G\left(\frac{\delta_1 y}{\left(1+\delta_2 y^2\right)^{\frac{1}{2}}}\right) \quad (7)$$

where, $\delta_1 \in R$, $\delta_2 \ge 0$, g and G are univariate Laplace density function with mean θ, variance S and the distribution function of univariate standard normal respectively. Also, a d-dimensional random vector $X \in R^d$ has a multivariate skew Laplace normal distribution denoted by ($X \sim MSLN_d(\theta, S, \delta)$ if its probability density function is given by:

$$\psi_d(x;\theta,S,\delta) = 2f_{ML}(x;\theta,S)\Phi\left(\delta'\tilde{S}^{-1}(x-\theta)\right) \quad (8)$$

where, $\theta \in R^d$ is a location parameter, a scale matrix $S \in R^{d\times d}$ is a positive definite, a skewness vector is $\delta \in R^d$, $\tilde{S} = \text{diag}(S_{11}, S_{22},..., S_{dd})^{½}$, $S = (s_{ij})$, I, j = 1,2,...,d and $f_{ML}(X; \theta, S)$ is a probability density function of a multivariate Laplace distribution.

$$f_{ML}(x;\theta,S) = \frac{\det(S)^{-\frac{1}{2}}}{2^d(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\exp\left\{-\sqrt{(x-\theta)' S^{-1}(x-\theta)}\right\} \quad (9)$$

The stochastic representation of $X \sim MSLN_d(\theta, S, \delta)$ can be obtained as a mixture of multivariate normal distribution $U_1 \sim MN_d(\theta, S)$ and univariate standard normal distribution $U_2 \sim N(0, 1)$:

$$Y = \theta + (S)^{\frac{1}{2}}\left[\frac{\delta|U_2|}{\sqrt{\upsilon(\upsilon+\delta'\delta)}} + \left(\upsilon I_d + \delta\delta'\right)^{\frac{1}{2}}U_1\right] \quad (10)$$

where, υ has the inverse gamma distribution with the probability density function:

$$g(\upsilon) = \frac{1}{2^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}\upsilon^{-\left(\frac{d+1}{2}+1\right)}e^{-\frac{1}{\upsilon}} \quad (11)$$

and suppose that U_1, U_2 and υ are mutually independent. For the details about this see. Arslan[15] shows that The conditional distribution of X given Y will be normal distribution with mean $\theta+Z^{-1}\delta$ and variance $Z^{-1}S$.

The joint density function of X and Z is:

$$f(x,z) = \frac{\det(S)^{-\frac{1}{2}}\exp\left\{(x-\theta)' S^{-1}(x-\theta)\right\}}{2^d\sqrt{(1+\delta' S^{-1}\delta)}(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}z^{-\frac{3}{2}} \times\exp\left\{-\frac{1}{2}\left((x-\theta)' S^{-1}(x-\theta)z + (1+\delta' S^{-1}\delta)z^{-1}\right)\right\} \quad (12)$$

And the conditional density function of Z given X is:

$$f(z|x) = \frac{\sqrt{(1+\delta' S^{-1}\delta)}}{\sqrt{2\pi}}\exp\left\{\begin{matrix}-\sqrt{(1+\delta' S^{-1}\delta)}z^{-\frac{3}{2}}\sqrt{(x-\theta)' S^{-1}(x-\theta)}\\ -\frac{1}{2}\left((x-\theta)' S^{-1}(x-\theta)z + (1+\delta' S^{-1}\delta)z^{-1}\right)\end{matrix}\right\} \quad (13)$$

We note clearly that when δ = 0 then the multivariate skew Laplace distribution can be reduced to the symmetric multivariate Laplace distribution. The mean vector and covariance matrix of X are derived by Doğruand Arslan and other authors in the following form:

$$E(X) = \theta + \sqrt{\frac{2}{\pi}}(S)^{\frac{1}{2}}\delta\xi \quad (14)$$

$$Var(X) = (d+1)S - \frac{2}{\pi}(S)^{\frac{1}{2}}\delta\delta'(S)^{\frac{1}{2}}\xi^2 \quad (15)$$

Where:

$$\xi = E\left(\frac{(\upsilon)^{-\frac{1}{2}}}{\sqrt{\upsilon+\delta'\delta}}\right)$$

which, can be computed by using numerical methods such as Monte Carlo and importance sampling methods.

The characteristic function of a random vector $X \sim MSLN_d(\theta, S, \delta)$ is given by:

$$\Psi_X(r) = \exp\left(\frac{ir'\theta}{2}\right)E_\upsilon\left(\exp\left(\frac{-\upsilon^{-1}r'Sr}{2}\right)\left(1+i\tau\kappa'\omega r\right)\right) \quad (16)$$

Where:

$$\kappa = \frac{\upsilon^{-\frac{3}{2}}\tilde{S}\delta}{\left(1+\upsilon^{-2}\delta'\tilde{S}\delta\right)^{\frac{1}{2}}}$$

$\tilde{S} = \omega S\omega$ and $\omega = diag(s_{11}, s_{22},\ldots,s_{dd})^{1/2}$, $S = (S_{ij})$, I, j = 1, 2,…,d and the function $\tau(\cdot)$ is defined as follows:

$$\tau(y) = \int_0^y \sqrt{\frac{2}{\pi}}\exp\left(\frac{u^2}{2}\right)du, \quad y>0, \tau(-y) = -\tau(y) \quad (17)$$

We note that the expectation given in the equation 16 can be calculated by numerical methods.

Entropies

Proposition 4: Let $X_0 \sim ML_d(\theta, S)$. Then the Shannon and Rényi entropies of X_0 are given in the following forms, respectively

$$H(x_0;\theta,S) = -2\ln\left(\frac{\det(S)^{-\frac{1}{2}}}{2^d\left(1+\delta'S^{-1}\delta\right)(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right) - 2d \quad (17)$$

$$R_\alpha(x_0;\theta,S) = \ln\left[\frac{\alpha^{\frac{-d}{1-\alpha}}\det(4\pi S)^{\frac{1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}}\right], 0<\alpha<\infty, \alpha\neq 1 \quad (18)$$

Lemma 2: Let $X \sim MSLN_d(\theta, S, \delta)$ distributed in the equation 8, Then:

$$E\left[\ln\left(\Phi\left(\delta'\tilde{S}^{-1}(x-\theta)\right)\right)\right] = \sqrt{\frac{2}{\pi}}\delta'S^{-\frac{1}{2}}\delta\xi \quad (19)$$

$$E\left[\left(\ln\left(f_{ML}(x;\theta,S)\right)\right)\right] = \ln\left(\frac{\det(S)^{-\frac{1}{2}}}{2^d\sqrt{\left(1+\delta'S^{-1}\delta\right)}(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right)(d+1)\delta'S^{-1}\delta + d \quad (20)$$

Proof: Directly:

$$\begin{aligned} E\left[\ln\left(\Phi\left(\delta'\tilde{S}^{-1}(x-\theta)\right)\right)\right] &= E\left[(x-\theta)'S^{-1}\delta\right] \\ &= E\left(Tr\left(\delta'S^{-1}(x-\theta)\right)\right) \\ &= Tr\left(\delta'S^{-1}E(x-\theta)\right) \\ &= Tr\left(\sqrt{\frac{2}{\pi}}\delta'S^{-\frac{1}{2}}\delta\xi\right) \\ &= \sqrt{\frac{2}{\pi}}\delta'S^{-\frac{1}{2}}\delta\xi \end{aligned}$$

Now, to prove part ii:

$$\begin{aligned} E\left[\left(\ln\left(f_{ML}(x;\theta,S)\right)\right)\right] &= \ln\left(\frac{\det(S)^{-\frac{1}{2}}}{2^d(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right) \\ &+ E\left\{\ln\left(-\sqrt{(x-\theta)'S^{-1}(x-\theta)}\right)\right\} \end{aligned}$$

If we use the conditional density function in equation 13 then the conditional expectation of Z^{-1} given X can be obtained as follows:

$$E\left(z^{-1}|x\right) = \frac{1}{\left(1+\delta'S^{-1}\delta\right)}\left(1-\sqrt{(x-\theta)'S^{-1}(x-\theta)}\right)$$

Where:

$$Z^{-1} \sim G\left(\frac{d+1}{2}, \frac{1}{2}\right)$$

Taking expectation for both sides of above equation, we have:

$$E\left(E\left(z^{-1}|x\right)\right) = \frac{1}{\left(1+\delta'S^{-1}\delta\right)}\left(1+E\left[-\sqrt{(x-\theta)'S^{-1}(x-\theta)}\right]\right)$$

Therefore:

$$E\left(z^{-1}\right) = \frac{1}{\left(1+\delta'S^{-1}\delta\right)}\left(1+E\left[-\sqrt{(x-\theta)'S^{-1}(x-\theta)}\right]\right)$$

But:

$$Z^{-1} \sim G\left(\frac{d+1}{2}, \frac{1}{2}\right)$$

Then:

$$d+1=\frac{1}{\left(1+\delta^{'}S^{-1}\delta\right)}\left(1+E\left[-\sqrt{\left(x-\theta\right)^{'}S^{-1}\left(x-\theta\right)}\right]\right)$$

Consequently:

$$E\left[\left(\ln\left(f_{ML}\left(x;\theta,S\right)\right)\right)\right]=\ln\left(\frac{\det\left(S\right)^{-\frac{1}{2}}}{2^{d}\left(\pi\right)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right)+\left(d+1\right)\delta^{'}S^{-1}\delta+d$$

Proposition 5: Let $X_0 \sim ML_d(\theta, S)$ and $X \sim MSLN_d(\theta, S, \delta)$. Then the Shannon entropy can be written as follows:

$$H\left(X;\theta,S,\delta\right)=H\left(X_0;\theta,S\right)-\hat{C}_{\tilde{\delta}} \quad (21)$$

Where:

$$\hat{C}_{\tilde{S}}=2\left(d+1\right)\delta^{'}S^{-1}\delta+\sqrt{\frac{2}{\pi}}\delta^{'}S^{-\frac{1}{2}}\delta\xi \quad (22)$$

Proof: By taking the natural logarithm and expectation for both sides of equation 8, we have:

$$H\left(X;\theta,S,\delta\right)=-2E\left(\ln\left(f_{ML}\left(x;\theta,S\right)\right)\right)-E\left(\ln\left(\Phi\left(\delta^{'}\tilde{S}^{-1}\left(x-\theta\right)\right)\right)\right)$$

Using lemma 2., we obtain:

$$H(X;\theta,S,\delta)=-2\ln\left(\frac{\det\left(S\right)^{-\frac{1}{2}}}{2^{d}\sqrt{\left(1+\delta^{'}S^{-1}\delta\right)}\left(\pi\right)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right)-2d-2\left(d+1\right)\delta^{'}S^{-1}\delta-\sqrt{\frac{2}{\pi}}\delta^{'}S^{-\frac{1}{2}}\delta\xi$$

From proposition 4., we have:

$$H\left(X;\theta,S,\delta\right)=H\left(X_0;\theta,S\right)-2\left(d+1\right)\delta^{'}S^{-1}\delta-\sqrt{\frac{2}{\pi}}\delta^{'}S^{-\frac{1}{2}}\delta\xi$$

Lemma 3:. Let $X \sim MSLN_d(\theta, S, \delta)$. Then:

$$\int_{R^d}\left(\psi_d\left(x;\theta,S,\delta\right)\right)^{\alpha}dx=\left(\frac{\alpha^{-\frac{d}{1-\alpha}}\det\left(4\pi S\right)^{\frac{1}{2}}\Gamma\left(\frac{d+1}{2}\right)\sqrt{\left(1+\delta^{'}S^{-1}\delta\right)}}{\sqrt{\pi}}\right)^{(1-\alpha)} \quad (23)$$

Proof: From equation 8, we get:

$$\int_{R^d}\left(\psi_d\left(x;\theta,S,\delta\right)\right)^{\alpha}dx=\int_{R^d}\frac{\det\left(S\right)^{-\frac{1}{2}\alpha}}{2^{\alpha(d-1)}\left(\pi\right)^{\frac{1}{2}\alpha(d-1)}\left(\Gamma\left(\frac{d+1}{2}\right)\right)^{\alpha}}\exp\left\{-\alpha\sqrt{\left(x-\theta\right)^{'}S^{-1}\left(x-\theta\right)}\right\}\left(\Phi\left(\delta^{'}\tilde{S}^{-1}\left(x-\theta\right)\right)\right)^{\alpha}dx$$

Replacing $S^{-1/2}\delta$ by $\tilde{\delta}$, $\alpha^{-2}S$ by S_α and using the change of variables $y = S_\alpha^{-1/2}(X-\theta)$ associated with Jacobian matrix $S^{1/2}$ in above equation, we get:

$$\int_{R^d}\left(\psi_d\left(x;\theta,S,\delta\right)\right)^{\alpha}dx=\frac{\alpha^{-\alpha d}\det\left(S_{\alpha}\right)^{-\frac{1}{2}\alpha+\frac{1}{2}}}{2^{\alpha d}\left(\pi\right)^{\frac{1}{2}\alpha(d-1)}\left(\Gamma\left(\frac{d+1}{2}\right)\right)^{\alpha}}\int_{R^d}\exp\left\{-\sqrt{y^{'}}y+y^{'}\tilde{\delta}\right\}dy$$

where, $Y \sim ML_d(0, I_d, \tilde{\delta})$. This complete the proof.

Corollary 1: If $X_0 \sim ML_d(\theta, S)$ and $X \sim MSLN_d(\theta, S, \delta)$., then the Rényi entropy can be written as:

$$R_\alpha(X; \theta; S, \delta) = R_\alpha(X_0; (\theta, S, \delta) + K_\delta \quad (24)$$

Where:

$$K_\delta = \ln\left(\sqrt{\left(1+\delta^{'}S^{-1}\delta\right)}\right)$$

Proof: Taking natural logarithm and multiplying by $\frac{1}{1-\alpha}$ or both sides of equation 23, we get:

$$R_\alpha\left(X;(\theta,S,\delta)\right)=\frac{1}{1-\alpha}\ln\left(\frac{\alpha^{-\frac{d}{1-\alpha}}\det\left(4\pi S\right)^{\frac{1}{2}}\Gamma\left(\frac{d+1}{2}\right)\sqrt{\left(1+\delta^{'}S^{-1}\delta\right)}}{\sqrt{\pi}}\right)^{(1-\alpha)}$$

Hence:

$$R_\alpha\left(X;(\theta,S,\delta)\right)=\ln\left(\frac{\alpha^{-\frac{d}{1-\alpha}}\det\left(4\pi S\right)^{\frac{1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}}\right)+\ln\left(1+\delta^{'}S^{-1}\delta\right)$$

Proposition 4: Gives us the result of this corollary.

It is easy to observe that skew Laplace normal Shannon entropy is obtained from 21 by taking the limit as α converges to 1.

Computational Implementation and Numerical Simulations: All numerical computations were made with MATLAB 2020a. The integrals of skew *Laplace normal* Shannon entropy of Equation 21 were evaluated using numerical methods such as Monte Carlo and importance sampling methods. This section illustrates the relationship between parameters α and δ with entropy for d = 1, 2, 3 and 4 dimensions. Consider $X \sim MSLN_d(\theta, S, \delta)$ with the following parameters:

- Case 1: d = 1, θ = 0.9, S = 0.5, δ = 0.5
- Case 2: $d=2, \theta=\begin{pmatrix}1\\2\end{pmatrix}, S=\begin{pmatrix}0.7 & 0.1\\0.1 & 1\end{pmatrix}, \delta=\begin{pmatrix}0.5\\1\end{pmatrix}$

- Case 3: $d=3,\ \theta=\begin{pmatrix}0\\1\\0\end{pmatrix},\ S=1_3,\ \delta=\begin{pmatrix}0.5\\1\\0.4\end{pmatrix}$
- Case 4: $d=3,\theta=\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\ S=1_4,\ \delta=\begin{pmatrix}0.5\\1\\0.4\\0.1\end{pmatrix}$
- Case 5: $d=3,\theta=\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix},\ S=1_5,\ \delta=\begin{pmatrix}0.5\\1\\0.4\\0.1\\1\end{pmatrix}$

For each case in above example, Table 1. Summarizes the values of entropy of $X\sim MSLN_d(\theta, S, \delta)$ such that α = 1-5, 10, 20 and α converges to infinite value. The following table shows that there is a relationship between the values of Rényi entropy and the values of the parameters α and d whereas, for any case in this example the Rényi entropy converges to finite value as $\alpha\to\infty$. The dispersion matrix S plays an important role in Rényi entropy, mainly by the increment of matrix dimension related to d. Moreover, the Rényi entropy decreases when *α* increases, as mentioned in the properties of $R_\alpha(X)$.

Figure 1 presents the cases mentioned in example 1. we observe that the Rényi entropy of $X\sim MSLN_d(\theta, S, \delta)$ converges to a finite value for any value of α. it can be seen that the Shannon and Rényi entropies increases for increasing the values of d. The determine of dispersion matrices play important role in determining the value of Rényi entropy where it increases with its increasing. Also, we note that the effectiveness of α on Rényi entropy is slow whenever the value of α is large and vice versa when it is small.

Application to mixture of multivariate skew *Laplace normal* distributions: Let us consider the definitions of Arslan[15] for Mixture of Multivariate Skew Laplace normal distributions. The probability density function of an m-component mixture model with parameter vector set (θ, S, δ); $\theta = \{\theta_1, \theta_2,...,\theta_m\}$ a set of vectors represent location parameters, $S = \{S_1, S_2,...,S_m\}$ a set of dispersion matrices, the shape vector parameter $S = \{\delta_1, \delta_2,...,\delta_m\}$ is:

$$f(x;\theta,S,\delta,\varepsilon)=\sum_{i=1}^{m}\varepsilon_i f(x;\theta_i,S_i,\delta_i) \quad (25)$$

where, ε_i denotes the mixing probability with $\varepsilon_i\geq 0$, $\sum_{i=1}^{n}\varepsilon_i=1$ $f(x; \theta_i, S_i, \delta_i)$ are defined as in equation 8 for a known $(\theta_i, S_i, \delta_i)$, i = 1,...,m and the notation MMSLNd (θ, S, δ, ε) represents mixture of multivariate skew Laplace distribution.

If $\kappa = (\kappa_1, \kappa_2,...k_m)$ represent a set of n latent allocations for densities of observations×then:

$$f(x;\theta,S,\theta,\varepsilon)=\prod_{j=1}^{m}f(x;\theta,S,\delta,\kappa_j)$$

where, $Pr(\kappa_j = i|\varepsilon) = \varepsilon_i$ then for any j-th component density in equation 8 is obtained as:

$$X_j\left|(\kappa_j=i)\right.\overset{d}{=}\theta_i+(S_i)^{\frac{1}{2}}\left[\frac{\delta_i|U_{2j}|}{\sqrt{\upsilon_i(\upsilon_i+\delta_i'\delta_i)}}+(\upsilon_i I_d+\delta_i\delta_i')^{\frac{1}{2}}U_{1j}\right],\ j=1,2,\ldots,m \quad (26)$$

where, $U_{1j}\sim MN_d(\theta_j, S_j)$, $U_{2j}\sim N(0, 1)$ and for each J = 1, 2,..m, υ_j has the inverse gamma distribution with the probability function which defined in equation 11. Also, suppose that U_{1j}, U_{2j} and υ_j are mutually independent.

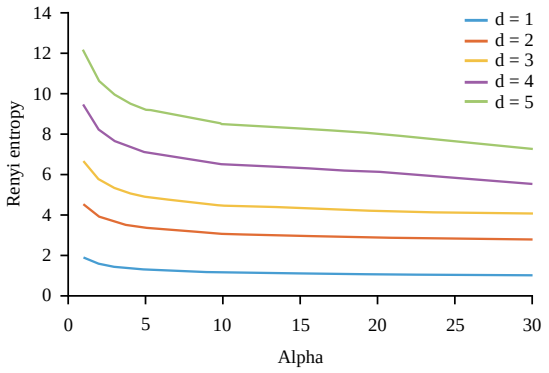


Fig. 1: The horizontal line represents the values of parameter α and the vertical line is the Rényi entropy of $X\sim MSL_d(\theta, S, \delta)$ with parameters in example 1

Table 1: The values of entropy of $MSLN_d(\theta; S, \delta)$ are computed for α = 2-5, 10, 20 and α converges to infinite of one to five dimensions

Case	Shannon entropy	$R_\alpha(x; \theta, S, \delta)$						
d	H(x; θ, S, δ)	α = 2	α = 3	α = 4	α = 5	α = 10	α = 20	α→∞
1	1.9250	1.6182	1.4743	1.3871	1.3274	1.1809	1.0827	1.0423
2	4.6138	4.0001	3.7124	3.5380	3.4185	3.1255	2.9291	2.8483
3	6.7775	5.8569	5.4254	5.1638	4.9845	4.5450	4.2505	4.1293
4	9.5977	8.3703	7.7950	7.4461	7.2072	6.6211	6.2284	5.6254
5	12.3598	10.8255	10.1063	9.6703	9.3716	8.6390	8.1482	7.3604

Equations 14-15 gives the first and second moments for each i-th component of X, respectively, the mean and variance of X can be obtained as follows:

$$E(X)=\sum_{i=1}^{m}\varepsilon_i\left(\theta_i+\sqrt{\frac{2}{\pi}}(S_i)^{\frac{1}{2}}\delta_i\xi_i\right) \tag{27}$$

$$\begin{aligned}Cov(X)=\sum_{i=1}^{n}\varepsilon_i\Bigg(&(d+1)S_i-\frac{2}{\pi}(S_i)^{\frac{1}{2}}\delta_i\delta_i'(S_i)^{\frac{1}{2}}\xi_i{}^2\\&+\left(\theta_i+\sqrt{\frac{2}{\pi}}(S_i)^{\frac{1}{2}}\delta_i\xi_i\right)\left(\theta_i+\sqrt{\frac{2}{\pi}}(S_i)^{\frac{1}{2}}\delta_i\xi_i\right)'\Bigg)-E(X)E(X)'\end{aligned} \tag{28}$$

Lemma 4: Let X~MMSLN$_d$(θ, S, δ, ε). Then the following inequality is accomplished:

$$C_{lower}\le H(X;\theta,S,\delta,\varepsilon)\le C_{upper} \tag{29}$$

Where:

$$C_{upper}=\frac{1}{2}\ln\left(\det\left(2\pi\exp(1)\mathcal{M}\right)\right) \tag{30}$$

$$\begin{aligned}C_{lower}=&\sum_{i=1}^{m}-2\varepsilon_i\ln\left(\frac{\det(S_i)^{-\frac{1}{2}}}{2^d\left(1+\delta_i'S_i^{-1}\delta_i\right)(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right)\\&-\sum_{i=1}^{m}\varepsilon_i\left(2(d+1)\delta_i'S_i^{-1}\delta+\sqrt{\frac{2}{\pi}}\delta_i'S_i^{-\frac{1}{2}}\delta_i\xi_i\right)-2d\end{aligned} \tag{31}$$

Where:

$$\mathcal{M}=\sum_{i=1}^{n}\varepsilon_i\left((d+1)S_i-\frac{2}{\pi}(S_i)^{\frac{1}{2}}\delta_i\delta_i'(S_i)^{\frac{1}{2}}\xi_i{}^2+\mathcal{K}_i\mathcal{K}_i'\right) \tag{32}$$

$$\mathcal{K}_i=\theta_i+\sqrt{\frac{2}{\pi}}(S_i)^{\frac{1}{2}}\delta_i\xi_i-E(X) \tag{33}$$

Proof: Since $\mathcal{M}$ is a covariance matrix of X then by using lemma 1., we get on upper bound:

$$H(X;\theta,S,\delta,\varepsilon)\le\frac{1}{2}\ln\left(\det\left(2\pi\exp(1)\mathcal{M}\right)\right)$$

Now, to find the lower bound of Shannon entropy. From the expression of mixture density in (25) the Shannon entropy can be written in the following form

$$H(X;\theta,S,\delta,\varepsilon)=-E\left(\ln\left(\sum_{i=1}^{m}\varepsilon_i f(x;\theta_i,S_i,\delta_i)\right)\right)$$

Since In (x) is a convex function then -In(x) is a concave. Therefore, by using Jensen's inequality, we obtain:

$$H(X;\theta,S,\delta,\varepsilon)\ge\sum_{i=1}^{m}\varepsilon_i H(X;\theta_i,S_i,\delta_i)$$

Hence, the Shannon entropy of each component is obtained from proposition 5. This complete the proof.

Lemma 5: If X~MMSLN$_d$(θ, S, δ, ε), then for any positive integer α the following inequality is hold:

$$R_\alpha(X;\theta,S,\delta,\varepsilon)\le\mathfrak{C}_{Upper} \tag{34}$$

Where:

$$\begin{aligned}\mathfrak{C}_{Upper}=\frac{1}{1-\alpha}\ln\Bigg\{&\exp\left((1-\alpha)R_\alpha(X;\theta_m,S_m,\delta_m)\right)+\sum_{i-1}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^\alpha\\&\times\begin{pmatrix}\exp\left((1-\alpha)R_\alpha(X;\theta_i,S_i,\delta_i)\right)\\-\exp\left((1-\alpha)R_\alpha(X;\theta_{i+1},S_{i+1},\delta_{i+1})\right)\end{pmatrix}\Bigg\}\end{aligned} \tag{35}$$

Proof: Mixture density in equation 25 implies that:

$$\left(f(x;\theta,S,\delta,\varepsilon)\right)^\alpha=\left(\sum_{i=1}^{m}\varepsilon_i f(x;\theta_i,S_i,\delta_i)\right)^\alpha$$

Applying lemma 5. when p = α, we obtain:

$$\begin{aligned}\left(\sum_{i=1}^{m}\varepsilon_i f(x;\theta_i,S_i,\delta_i)\right)^\alpha\ge&\, f(x;\theta_m,S_m,\delta_m)^\alpha\\&+\sum_{i=1}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^\alpha\begin{pmatrix}\left(f(x;\theta_i,S_i,\delta_i)\right)^\alpha\\-\left(f(x;\theta_{i+1},S_{i+1},\delta_{i+1})\right)^\alpha\end{pmatrix}\end{aligned}$$

Taking the integral for both sides of above inequality over R^d, we obtain:

$$\begin{aligned}\int_{R^d}\left(f(x;\theta,S,\delta,\varepsilon)\right)^\alpha dx\ge&\int_{R^d}f(x;\theta_m,S_m,\delta_m)^\alpha dx+\sum_{i=1}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^\alpha\\&\int_{R^d}\begin{bmatrix}\left(f(x;\theta_i,S_i,\delta_i)\right)^\alpha\\-\left(f(x;\theta_{i+1},S_{i+1},\delta_{i+1})\right)^\alpha\end{bmatrix}dx\end{aligned}$$

Again, taking natural logarithm and multiplying by $\frac{1}{1-\alpha}$ for both sides of last inequality, we have:

$$\begin{aligned}&R_\alpha(X;\theta,S,\delta,\varepsilon)\\&\le\frac{1}{1-\alpha}\ln\left\{\int_{R^d}f(x;\theta_m,S_m,\delta_m)^\alpha dx+\sum_{i=0}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^\alpha\int_{R^d}\begin{bmatrix}\left(f(x;\theta_i,S_i,\delta_i)\right)^\alpha\\-\left(f(x;\theta_{i+1},S_{i+1},\delta_{i+1})\right)^\alpha\end{bmatrix}dx\right\}\end{aligned}$$

Lemma 6: Let X~MMSLN$_d$(θ, S, δ, ε). Then for any positive integers $k_1, k_2,\ldots,k_m$ such that $\sum_{i=1}^{m}k_i=\alpha$ the following approximation:

$$\frac{1}{\alpha}\ln\left\{\left(\frac{\alpha!}{k_1!k_2!\ldots k_m!}\right)\prod_{i=1}^{m}\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}\right\}\cong-\sum_{i=1}^{m}\gamma_i\ln\left(\frac{\gamma_i}{\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)}\right) \tag{36}$$

Is accomplished as $\alpha\leftarrow\infty$, where:

$$\gamma_i=\frac{k_i}{\alpha},i=1,2,\ldots,m$$

Proof:

$$\frac{1}{\alpha}\ln\left\{\left(\frac{\alpha!}{k_1!k_2!\ldots k_m!}\right)\prod_{i=1}^{m}\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}\right\}$$
$$=\frac{1}{\alpha}\ln(\alpha!)-\frac{1}{\alpha}\sum_{i=1}^{m}\ln(k_i!)+\frac{1}{\alpha}\sum_{i=1}^{m}k_i\ln\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)$$

By using the approximation of factorial in above equality, we have:

$$\frac{1}{\alpha}\ln\left\{\left(\frac{\alpha!}{k_1!k_2!\ldots k_m!}\right)\prod_{i=1}^{m}\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}\right\}$$
$$=\ln(\alpha)-1+\frac{1}{2\alpha}\ln(2\pi\alpha)-\frac{1}{\alpha}\sum_{i=1}^{m}k_I\ln(k_i)$$
$$+\frac{1}{\alpha}\sum_{i=1}^{m}k_i-\frac{1}{2\alpha}\sum_{i=1}^{m}\ln(2\pi k_i)+\sum_{i=1}^{m}\gamma_i\ln\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)$$

But $\gamma_i=\frac{k_i}{\alpha},i=1,2,...,m$, then $\sum_{i=1}^{m}\gamma_i=1$, Consequently:

$$\frac{1}{\alpha}\ln\left\{\left(\frac{\alpha!}{k_1!k_2!\ldots k_m!}\right)\prod_{i=1}^{m}\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}\right\}$$
$$=\frac{1}{2\alpha}\left[\ln\left(\frac{(2\pi\alpha)^{1-n}}{\prod_{i=1}^{n}\gamma_i}\right)\right]-\sum_{i=1}^{m}\gamma_i\ln\left(\frac{\gamma_i}{\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)}\right)$$

But:

$$\mathrm{Lim}_{\alpha\to\infty}\sum_{i=1}^{m}\gamma_i\ln\left(\frac{\gamma_i}{\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)}\right)=0$$

This implies that the result in this lemma is satisfied.

Lemma 7: The approximation:

$$R_\alpha\left(X;\theta,S,\delta,\varepsilon\right)\cong\frac{1}{1-\alpha}\ln\left(\sum_{k_i\in B}\left(\prod_{i=1}^{m}\left(\gamma_i\right)^{-k_i}\right)\times\left(\prod_{i=1}^{m}\varepsilon_i^{\,k_i}\exp\left((1-\alpha)R_{k_i}\left(X;\theta_i,S_i,\delta_i\right)\right)\right)\right) \tag{37}$$

Is satisfied as $\alpha\to\infty$.
Where:

$$\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}=m^{\alpha},B=\left\{k_i\in N,k_i>0,\sum_{i-1}^{m}k_i=\alpha,i=1,2,\ldots,m\right\}$$

Proof: The integral of mixture model in equation 25 with exponent α can be written as:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}dx=\int_{R^d}\left(\sum_{i=1}^{m}\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{\alpha}$$

By using multinomial theorem, we obtain:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}dx=\int_{R^d}\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}dx \tag{38}$$

Where:

$$\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}=m^{a},B=\left\{k_i\in N,k_i>0,\sum_{i=1}^{m}k_i=\alpha,i=1,2,\ldots,m\right\}$$

By replacing right side of equation 38 in equation 36, we get:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}dx\cong\int_{R^d}\sum_{k_i\in B}\exp\left\{-\alpha\sum_{i=1}^{m}\gamma_i\ln\left(\frac{\gamma_i}{\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)}\right)\right\}dx$$

The last approximation can be written as:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}=\sum_{k_i\in B}\prod_{i=1}^{m}\left(\gamma_i\right)^{-k_i}\int_{R^d}\prod_{i=1}^{m}\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}dx$$

Consequently:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}=\sum_{k_i\in B}\left[\prod_{i=1}^{m}\left(\gamma_i\right)^{-k_i}\right]\left[\prod_{i=1}^{m}\int_{R^d}\left(\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}dx\right]$$

If we take the natural logarithm and multiplying by $\frac{1}{1-\alpha}$ for both sides of last approximation, then the proof is completed:

Lemma 8: Consider X~MMSLN$_d$(θ, S, δ, ε), then for any positive integer α the lower bound of Rényi entropy appears as follow:

$$R_\alpha\left(X;\theta,S,\delta,v,\varepsilon\right)\geq\mathfrak{C}_{\mathrm{Lower}} \tag{39}$$

Where:

$$\mathfrak{C}_{\mathrm{Lower}}=\frac{1}{1-\alpha}\ln\left(\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i\right)^{k_i}\times\exp\left\{\frac{(1-\alpha)}{\alpha}\sum_{i=1}^{m}k_iR_\alpha\left(X;\theta_i,S_i,\delta_i\right)\right\}\right) \tag{40}$$

Proof: The Rényi entropy of X~MMSLN$_d$(θ, S, δ, ε) can be written in the following form:

$$R_\alpha\left(X;\theta,S,\delta,\varepsilon\right)=\frac{1}{1-\alpha}\ln\left(\int_{R^d}\left(\sum_{i=1}^{m}\varepsilon_i f\left(x;\theta_i,S_i,\delta_i\right)\right)^{\alpha}dx\right)$$

By using multinomial theorem in Abid, we obtain:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}dx=\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i\right)^{k_i}\int_{R^d}\prod_{i=1}^{m}\left(f\left(x;\theta_i,S_i,\delta_i\right)\right)^{k_i}dx$$

Applying generalized HÖlder's Inequality for m-products, we get:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}dx\le\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i\right)^{k_i}\prod_{i=1}^{m}\left(\int_{R^d}\left(f\left(x;\theta_i,S_i,\delta_i\right)\right)^{p_ik_i}dx\right)^{\frac{1}{p_i}}$$

Wherein, $p_1, p_2,...,p_m$ and:

$$\sum_{i=1}^{m}\frac{1}{p_i}=1$$

The last equation can be written as:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}dx\le\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i\right)^{k_i}\exp\left\{\sum_{i=1}^{m}\left(\frac{\left(1-p_ik_i\right)}{p_i}R_{p_ik_i}\left(X;\theta_i,S_i,\delta_i\right)\right)\right\}$$

We choose $p_i=\frac{\alpha}{k_i},i=1,2,...,m,$ such that:

$$\sum_{i=1}^{m}\frac{1}{p_i}=\sum_{i=1}^{m}\frac{k_1}{\alpha}=1$$

And $1\le\frac{\alpha}{k_i}\le\alpha$ to obtain:

$$\int_{R^d}\left(f\left(x;\theta,S,\delta,\varepsilon\right)\right)^{\alpha}dx\le\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i\right)^{k_i}\exp\left\{\frac{\left(1-\alpha\right)}{\alpha}\sum_{i=1}^{m}k_iR_{\alpha}\left(X;\theta_i,S_i,\delta_i\right)\right\}$$

We complete the proof by taking the natural logarithm and multiplying by $\frac{1}{1-\alpha}$ for both sides of last inequality.

Theorem 1: Let $X\sim MMSLN_d(\theta, S, \delta, \varepsilon)$. Then the approximate Renyi entropy of X appears as follow:

$$\begin{aligned}R_{\alpha}\left(X;\theta,S,\delta,\varepsilon\right)=\frac{1}{2\left(1-\alpha\right)}\Bigg\{&\ln\Bigg(\sum_{k_i\in B}\frac{\alpha!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i\right)^{k_i}\\&\times\exp\left\{\frac{\left(1-\alpha\right)}{\alpha}\sum_{i=1}^{m}k_iR_{\alpha}\left(X;\theta_i,S_i,\delta_i\right)\right\}\Bigg)\\&+\left(1-\alpha\right)R_{\alpha}\left(X;\theta_m,S_m,\delta_m\right)\\&+\ln\Bigg(\sum_{i=1}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^{\alpha}\\&\times\begin{pmatrix}\exp\left(\left(1-\alpha\right)R_{\alpha}\left(X;\theta_i,S_i,\delta_i\right)\right)\\-\exp\left(\left(1-\alpha\right)R_{\alpha}\left(X;\theta_{i+1},S_{i+1},\delta_{i+1}\right)\right)\end{pmatrix}\Bigg)\Bigg\}\end{aligned}\tag{41}$$

Proof: The proof is directed from lemmas 11 and 14., by taking the mean of upper and lower bounds.

CONCLUSION

We derive upper and lower bounds on the entropy in both types (Shannon and Rényi) of a multivariate skew Laplace normal random variable. Then, we extended these tools to the class of finite mixture of multivariate skew Laplace normal densities. Considering the average of these bounds, the approximate value of entropy can be calculated. Both entropies converge to finite value of a multivariate skew Laplace normal random variable and its mixture model for any values of α order and dimension d. Given that mixture skew Laplace normal entropies are localized between the upper and lower bounds, the average of these bounds can be used as an approximation of the mixture skew Laplace normal entropies. In addition, the mixture skew Laplace normal entropy bounds provide useful information about the data and could be considered as a criterion to choose the possible number of components in each gender-based group. Finally, we encourage researchers to use the proposed approach for real-world applications and data analysis, such as environmental and biological data[16].

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