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Age-Structured Transmission Dynamics Model for Vertical and Horizontal Transmission of HIV/AIDS

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Abstract: Transmission dynamics model for HIV/AIDS, along the line of Mckendrick-Forester age-structured model is proposed with the natural mortality rate and the fertility functions assumed to be age depended, similar to Doma, Gurtin-MacCamy definitions. The solutions to the governing equations are obtained and the steady states are examined for their local stability. The model is further extended to study the case of constant mortality rate and an exponential type of interaction function. It is observed that the endemic steady exist and asymptotically stable.

Key words: Steady state, HIV/AIDS, mortality rate, interaction function, infection rate, AIDS incubation period, HIV progression rate, HIV infectious period

INTRODUCTION

Expositions on the origin, Biological and transmission mechanism of HIV/AIDS is in Castillo-Chavez *et al*. (1991) and Hethcotte and Ark (1992). However, there are three known modes of transmission of HIV. These are sexual contact with an infected person, direct contact with HIV-infected blood or fluid and lastly, transmission from an infected mother to her child, called Mother Tto Child Transmission (MTCT). However, in this research we intend to examine both heterosexual and mother to child modes of transmission of HIV using existing model assumptions of proportionate mixing in an age-structured epidemic models with a proposed exponential contact function.

MODEL PARAMETERS AND FORMULATION

$\mu(a, \delta(t))$ = Natural mortality rate
$r(a)$ = Progression rate from HIV-infection to AIDS
$\beta(t, a)$ = Force of infection
α: (a) = Disease induced mortality rate
ε = Probability of infecting the new born with HIV-virus
$S(t, a)$ = Population size of the Susceptible compartment of age a
$I(t, a)$ = Population size of the infected compartment of age a
$A(t, a)$ = Population size of AIDS, compartment of age a

We assume proportionate mixing of the population and that HIV-infected are not noticeable and are sexually productive. They can give birth to new born. Also assume that AIDS-infected are not reproductive, easily noticeable and not sexually interacted with. So that heterosexual contact with AIDS-infected person is avoidable, for fear of infection. Since, there is no care, for AIDS any infected will surely die from the diseases, (Olowofeso and Weama, 2005). We assume that the natural mortality and fertility rates are influenced by age in line with Doma (2004), assumption of age vital rates. Suppose, the heterosexual active and interacting number of adult satisfies the following,

$$N(t, a) = S(t, a) + I(t, a) + A(t, a)$$

Then, the dynamics of the population compartment can be described by the following partial differential equations,

$$S_t(t, a), S_a(t, a) = -[\mu(a, \delta(t, a)) + \beta(t, a)] S(t, a) \quad (1)$$

$$I_t(t,a), I_a(t,a) = \beta(t,a) S(t,a) - [r(a) - \mu(a, \delta(t,a)] I(t,a) \quad (2)$$

$$A_t(t, a), A_a(t, a) = r(a) I(t, a) - \alpha(a) A(t, a) \quad (3)$$

$$S(0, a) = S_a(a), I(0, a) = I_0(a), A(0, a) = A_0(a)$$
$$N(0, a) = N_0(a),$$

$$S(t,0) = \int_0^\infty \eta(a,)[S(t,a) + (1-\varepsilon)I(t,a)]da \quad (4)$$

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$$I(0,t) = \varepsilon \int_0^{\infty} \eta(a,)I(t,a)da \quad (5)$$

$$\beta(t,a) = \frac{\int_0^{\infty} k(a,\bar{a})I(t,\bar{a})d\bar{a}}{\int_0^{\infty} N(t,a)da} \quad (6)$$

$$A(t, a) = 0$$

Where, k (a, ā) is the interaction coefficient, defined as the probability that a susceptible of age a interacts with an infected individual of age ā and become infected, N (t) is the total population of age a at time t. The total number of susceptible and infected of age a are defined as,

$$S(t) = \int_0^{\infty} S(t,a)da,\ I(t) = \int_0^{\infty} I(t,a)da,$$

$$P(t) = \int_0^{\infty} N(t,a)da$$

The numbers of new born that are not HIV -infected per unit of time is, S (0, t) and the number of infected new born that are HIV-infected through MTCT transmission is I (t, 0). This means that all new born from susceptibles are susceptible, but a fraction ε of new born from infected parents are infected, through MTCT.

The natural mortality rate μ (a) is assumed the same for all susceptible and infectives and is a continuous function of age and the disease induced death rate is assumed non-negative continuous function of age a ϵ (0, ∞). Let N (a), 0≤ a≤ τ, (τ is the maximum age) be the density with respect to age of the total number of individuals, where the population is in a stationary demographic state, with equilibrium age density,

$$N(a) = \mu^* N \exp(-\int_0^{\infty} \mu(\varsigma, \delta(t,\varsigma))d\xi$$

N = The equilibrium population size
μ* = The crude death rate satisfying

$$\mu^* \int_0^{\infty} \nu(a)da = 1$$

$$\nu(a) = \exp(-\int_0^{\infty} \mu(\varsigma, \delta(t,\varsigma))d\varsigma$$

is the survival function, defined as the proportion of individuals who survive to age a. The stationary population size at age can be represented as N (a) = μ* N v (a), with life expectancy

$$L = \int_0^{\infty} \nu(a)da,$$

in line with Castillo-Chavez (1991). The basic reproductive number of the infection is defined as,

$$R_{\nu} = \int_0^{\infty} \eta(a,)\ \nu(a)da$$

Defining m (t, a) = S (t, a) + I (t,a) +A (t, a) and adding the equations we obtained the mckendrick-foerster age-structured model,

$$m_t(t,a) + m_a(t,a) = -\mu(a,\delta(a))m(t,a)$$
$$m(0,t) = \int_0^{\infty} \eta(a)m(t,a)da$$
$$m(0,a) = S_0(a) + I_0(a) + A_0(a)$$

The equation describing the disease dynamics, in line with Mckendrick-Forester, MacCamy and Gurtin, representation of the vital rates are,

$$S_t(t, a) + S_a(t, a) = -[\mu(a) + \delta(a)) + \beta(t, a)]\ S(t, a)$$

$$I_t(t, a) + I_a(t, a) = \beta(t, a)\ S(t, a) - [r(a) - \mu(a) + \mu(\delta(a))]\ I(t, a)$$

$$A_t(t, a) + A_a(t, a) = r(a)\ I(t, a) - \alpha(a)\ A(t, a)$$

$$S(0, a) = S_a(a),\ I(0, a) = I_0(a),\ A(0, a) = A_0(a)$$
$$N(0, a) = N_0(a)$$

$$S(t,0) = \int_0^{\infty} \eta(a)[S(t,a) + (1-\varepsilon)I(t,a)]da$$
$$I(0,t) = \varepsilon \int_0^{\infty} \eta(a)I(t,a)da$$
$$\beta(t,a) = \frac{1}{N(t)} \int_0^{\infty} k(a,\bar{a})I(t,\bar{a})d\bar{a}, p(t) = \int_0^{\infty} N(t,a)da$$
$$A(t,0) = 0$$

From Eq. 2.2, we define fraction of the susceptible, HIV-infectives and AIDS as,

$$s = \frac{S(t,a)}{N(a)},\ i(t,a) = \frac{I(t,a)}{N(a)},\ z(t,a) = \frac{A(t,a)}{N(a)} \quad (7)$$

This has the effect that the dynamics of the population vanish from the population, with only exception that the population age -density appears in the equation for the per capital infection rate (Inaba, 1990). Thus, the governing equations are,

$$\begin{aligned} s_t(t,a) + s_a(t,a) &= -\beta(t,a)s(t,a) \\ i_t(t,a) + i_a(t,a) &= \beta(t,a)s(t,a) - r(a)i(t,a) \\ z_t(t,a) + z_a(t,a) &= r(a)i(t,a) \_ \end{aligned} \quad (8)$$

$$S(0, a) = 1,\ i(0, a) = z(0, a) = 0$$

The per capital force of infection is,

$$\beta(t,a) = \int_0^\infty k(a,\bar{a})i(t,\bar{a})d\bar{a}$$

in line with Cushing (1994).

In this research, we consider, this system with the initial conditions, $s(a, 0) = s_0(a)$, $i(a, 0) = i(a)$, $z(a, 0) = z_0(a)$, where, $s_0(a) + i_0(a)$, $z_0(a) = 1$ and $r(a) = r$, for all individuals infected in I, compartment. Thus, for all $t \geq 0$, we have

$$s(t, a) + I(t, a) + z(t, a) = 1$$

The solution to the governing equation can be obtained via characteristic line,

$$\frac{da}{dt} = 1$$

on which

$$\frac{ds(t,a)}{dt} = -\beta(t,a)s(t,a) \text{ and}$$

$$\frac{di(a)}{dt} = \beta(a)s(t,a) - ri(t,a)$$

Where, a, s (t, a) and i (t, a) are defined by,

$$a = \begin{cases} t + a_0, & a > t \\ t - t_0, & a < t \end{cases}$$

$$s(t,a) = s(0,a_0)\exp\{-\int_{a_0}^a \beta(s)ds\},\ a > 0$$

using $s(0, a\text{-}t) = s(0,a_0) = s(a-t)$,

$$s(t,a) = \begin{cases} s(a-t)\exp\{-\int_{a-t}^a \beta(s)ds\},\ a > t \\ s(t_0,0)\exp\{-\int_0^a \beta(s)ds\},\ a < t \end{cases}$$

$$i(t,a)\{ \begin{array}{l} i(0,a\text{-}t)e^{-rt} + e^{-rt}\int_{a-t}^a \beta(\tau)s(t,\tau)d\tau,\ a > t \\ i(t_0,0)e^{-rt} + e^{-rt}\int_0^a \beta(\tau)s(t,\tau)d\tau,\quad a < t \end{array}$$

The renewal equations for the susceptible and infectrive population are,

$$\begin{aligned} b(t) &= s(t,0) = C^* - q\varpi(a)e^{-rt}, \\ \varpi(a) &= \phi + \int_0^\infty D(a)da\int_{a-t}^a \beta(t)s(a-t)\rho(a)dt,\ a > t \end{aligned}$$

Where,

$$\phi = \int_0^\infty D(a)i(0,a\text{-}t)da,$$

$$\rho(a) = \exp\{-\int_{a+t}^a \beta(t)dt\},\ D(a) = \eta(a)N(a)$$

$$b(t) = C^* - q\Pi(a)e^{-rt},$$

$$\Pi(a) = \int_0^\infty D(a)i(t_0,a)da + s(t_0,0)$$

$$\int_0^\infty D(a)da\int_0^a \beta(\tau)\exp\{-\int_0^a \beta(\tau)d\tau\}d\tau d\tau,$$

$$a < t$$

$$\begin{aligned} i(t,0) &= qe^{-rt}\varpi(a),\quad a > t \\ i(t,0) &= qe^{-rt}\Pi(a),\quad a < t \end{aligned}$$

However, for practical purposes, we considered $t > a$ and our expressions for s (t, a) i (t, a) and b (t) takes the following forms,

$$\begin{aligned} s(t,a) &= s(t_0,0)\exp\{-\int_0^a \beta(s)ds) \\ i(t,a) &= i(t_0,0)e^{-rt} + s(t_0,0)e^{-rt} \\ &\int_0^a \beta(s)m(s)ds,\ m(a) = \exp\{-\int_0^a \beta(s)ds\} \\ b(t) &= s(0,t) = C^* - q\Pi(a)e^{-rt} \\ i(t,0) &= q\Pi(a)e^{-rt} \end{aligned}$$

The limiting population size for the compartments and the birth rate b (t) when $t > a$ are,

$$\begin{aligned} s(t,a) &= s(t_0,0)\exp\{-\int_0^\infty \beta(s)ds\}, \\ \beta(s) &= \int_0^\infty k(s,\bar{s})i(t,\bar{s})d\bar{s} \\ b(t) &= s(t,0) = C^* \\ \text{and } i(t,0) &= 0 \end{aligned}$$

EXISTENCE AND LOCAL STABILITY OF DISEASE-FREE AND ENDEMIC STATE

Let $(s^*(a), i^*(a))^t$ be the endemic steady state solution for the Eq. (2) and β^* be the force of infection at

the steady and then the following steady state solution are obtained in line with Diekmann and Kretzahman (1991) and Doma (2004),

$$s^*(a) = D\exp\{-\beta^* a\},\ D = C^* - I^*(0) \tag{9}$$

$$\begin{aligned} &i^*(a) = (i^*(0) + \beta^* \psi(a))\exp\{-ra\},\\ &a > 0 \quad \psi(a) = \int_0^a e^{rv} s^*(v)dv \end{aligned} \tag{10}$$

$$\begin{aligned} &= i^*(0)e^{-ra} + \frac{\beta^* D}{\beta^* - r}e^{-\beta^* a} = (e^{-ra} - \frac{\beta^*}{\beta^* - r}e^{-\beta^* a})\\ &i^*(0) + \frac{\beta^* C^*}{\beta^* - r}e^{-\beta^* a},\ \beta^* \neq r \end{aligned} \tag{11}$$

Per capital force of infection at the steady state is,

$$\beta^* = \frac{i^*(0)\int_0^\infty k_2(a)N(a)da}{1 - \int_0^\infty \psi(a)N(a)e^{-ra}da} \tag{12}$$

Where,

$$\int_0^\infty \psi(a)N(a)e^{-ra}da \neq 1$$

The force of infection is zero when i* (0) is zero. In that case the disease-free steady state exists. Since vertical transmission of the disease is assumed, the infected new born into the population, i (0) ≠ 0 and the expression for infectious β* is a positive constant, satisfying,

$$\begin{aligned} &e^{-ra} - \frac{\beta^*}{\beta^* - r}e^{-\beta^* a} > 0\\ &\beta^* > \varphi(a)r, \end{aligned} \tag{13}$$

Where,

$$\varphi(a) = \frac{e^{-ra}}{e^{-ra} - e^{-\beta^* a}},\ e^{-ra} > e^{-\beta^* a}, \beta^* > r$$

Thus, the endemic steady state (s* (a), i* (a)) exists. Using the method in Castillo-Chavez *et al.* (1991) and Hethcote and Ark (1992) we substitute the infective population density i* (a) into the equation for the force of infection and obtained Lotka type characteristic equation,

$$\begin{aligned} &1 = \eta\int_0^\infty k_1(a)e^{-(ra+\phi(a))}(\int_0^a k_2(\bar{a})e^{-(\beta^* - r)\bar{a}}d\bar{a})da,\\ &\eta = s(0)\mu^* \end{aligned} \tag{14}$$

Which has a positive solution, β*, provided that the threshold condition is satisfied,

$$1 < \eta\int_0^\infty k_1(a)e^{-(ra+\phi(a))}(\int_0^a k_2(\bar{a})e^{r\bar{a}}d\bar{a})da = R_0 \tag{15}$$

Where, η = μ*s (0)

With the force of infection as positive solution of Eq. 14. When the threshold is satisfied the steady solution corresponds to an endemic steady state and when the threshold is not satisfied the force of infection is zero and the steady state corresponds to a disease-free steady state. To have a sustained endemic state for the disease, the infectious force should be greater than the progression rate from HIV to AIDS, as in Eq. 13. This will allow for long incubation period for AIDS, which is the target of HIV-infection intervention therapies. Since, r > 0, β* ≠ 0 and the steady state is an endemic steady state, with its local stability determined by linearization of the system in the immediate vicinity of the steady state. Consider the following perturbations about the steady state solution,

$$\begin{aligned} &s(t,a) = s^*(a) + \bar{s}(a)e^{pt}\\ &i(t,a) = i^*(a) + \bar{i}(a)e^{pt}\\ &\beta(t,a) = \beta^* \end{aligned} \tag{15}$$

Substitution into Eq. 4 an solving for, $\bar{s}$ (a) and $\bar{i}$ (a) gives the following equations

$$\bar{s}(a) = D(0)e^{-(p+r)a}$$

$$\bar{i}(a) = i^*(0)e^{-(p+r)a} + \beta^* D(0)\int_0^a e^{-[(\beta^* - r)\omega - (p+r)a]}d\omega$$

$$\beta^* = k_1(a)\mu^* \int_0^\infty k_2(a)\bar{i}(a)e^{-\phi(a)}da,$$

$$\phi(a) = \int_0^a \mu(a)da$$

$$s^*(0) = c - i^*(0),\ \ i^*(0) = mq$$

$$\int_0^\infty f(a)e^{-\phi(a)}i^*(a)da,\ m = \mu^* N,$$

$$D(0) = s(0) - s^*(0) =$$

$$-mq\int_0^\infty f(a)e^{-\phi(a)}i(a)da = -mqk, k = \int_0^\infty f(a)e^{-\phi(a)}i(a)da$$

Using the representation for i (a), we have,

$$\beta^* = i(0)\mu^* \int_0^\infty k_2(a)e^{-[(p+r)a+\phi(a)]}da + D(0)\mu^*$$

$$\beta^* \int_0^\infty k_1(a)e^{-[(p+r)a+\phi(a)]}(\int_0^\infty k_2(\bar{a})e^{-(\beta^* - r)\bar{a}}d\bar{a}$$

We obtained the Lotka type characteristic equation,

$$1 = \alpha\int_0^\infty k_1(a)e^{-[(\rho+r)a+\phi(a)]}(\int_0^a k_2(\bar{a})e^{-(\beta^*-r)\bar{a}}d\bar{a})da \quad (17)$$
$$= -mqk\mu^*, D(0) = -mqk$$

If all roots of the Lotka type characteristics equation have negative real part s, then all solutions of the form (9) (10) tend to zero as p tend to infinity. For infection force greater than zero, no non-negative solution can be found that will satisfy the Lokta characteristic equation.. However, for trivial steady state $\beta = 0$ Eq. (16), reduces to the form,

$$1 = \alpha\int_0^\infty k_1(a)e^{-[(\rho+r)a+\phi(a)]}(\int_0^a k_2(\bar{a})e^{r\bar{a}}d\bar{a})da \quad (18)$$

This will have a unique solution, $p \le 0$ and $p = 0$, only at the threshold.

To determine the local stability, of the steady of the endemic steady state, we propose the contact function,

$$k(a) = \begin{cases} e^{-a}, \text{ if } a < \tau \\ 0, \text{ if } a \ge \tau \end{cases} \quad (19)$$

Consistent with human sexual interaction behavior. That is the more an individual aged the less his activity level and also assumed natural mortality independed of age, μ say $\tau < \infty$, is the maximum age. Using Eq. (19) we find the characteristic equation and the value of β^*. We determine whether for these chosen values, the steady state, (9-10) is an asymptotically stable endemic steady state, in with the above expositions. From Eq. (18) we have,

$$\beta^* = \frac{-(2+\mu-r)}{2} \pm \frac{1}{2}\sqrt{(2+\mu-r)^2 - 4((1+\mu)r+\mu+\eta)} \quad (20)$$

Equation (20) is zero only when,

$$\frac{-(2+\mu-r)}{2} \text{ and } \frac{1}{2}\sqrt{(2+\mu-r)^2 - 4((1+\mu)r+\mu+\eta)}$$

are both zero, which is not possible, since μ, r and η are not zero. The infection function is not zero, the steady state Eq. 9 and 10 is not a disease-free steady state. Equation 9 and 10 is an endemic state only when the following holds,

$$(2+\mu+r)^2 + 4 < 8r + 6\mu r + 4\eta - \mu^2 - r^2,$$

Where $(2+\mu-r) < 0$

Thus, the roots are, $\beta^*_1 = \alpha_2$, $\beta^*_2 = \alpha_2$

Where,

$$\alpha_1 = -\frac{1}{2}(2+\mu-r) + \frac{1}{2}\sqrt{8r+6\mu r+4\eta - 4-\mu^2-r^2},$$
$$\alpha_2 = -\frac{1}{2}(2+\mu-r) - \frac{1}{2}\sqrt{8r+6\mu r+4\eta - 4-\mu^2-r^2} \quad (21)$$

From Eq. (17) we have,

$$p_1 = -\frac{\{1+(2+\alpha_1+\mu)(2+\alpha_1-r)}{(2+\alpha_1-r)}$$
$$p_2 = -\frac{\{1+(2+\alpha_2+\mu)(2+\alpha_2-r)}{(2+\alpha_2-r)} \quad (22)$$

Where,

$$1 = nqk\mu^*, \quad k = \int_0^a f(a)e^{\mu a}i(a)da$$

Since, p_1 are p_2 negative, Eq. (10, 11) will be reduced to an endemic disease state, for any fluctuations about the steady. The endemic steady exists and is asymptotically stable, for the assumed natural mortality rate and interaction function. However, since our target is the realization of the disease-free steady, where $\beta^* = 0$ This is achieved from Eq. (20) when,

$$r^2 - (8+6\mu)r - (4\eta - \mu^2 - 4) = 0 \quad (23)$$

Where,

$$r = 4 + 2\mu \pm \Delta, \ \Delta = \sqrt{9\mu^2 + 24\mu + 4\eta + 12} \quad (24)$$

Thus, for disease-free steady state,

$$r < \beta^* \quad (25)$$

While if the disease is endemic we would have $\beta^* < r$.

CONCLUSION

We have examined both heterosexual and mother to child transmission Dynamics of HIV-infection in a proportionate mixing population, with age depended natural mortality rate and extended it to study non-age depended mortality rate, in line with Gurtin and McCamy

age-structured population with exponential activity function. The studies shows that there exist an endemic steady state for HIV, given $\beta^* > r > 0$ and is locally asymptotically stable, However, for non-age depended natural mortality, with exponential interaction function, the endemic state is found to exist, if the following equations are satisfied,

$$(2 + \mu + r)^2 + 4 < 8r + 6\mu r + 4\eta - \mu^2 - r^2,$$

Where $(2 + \mu - r) < 0$

Is asymptotically stable if Eq. (22) holds. While for disease-free steady state Eq. (23-25) should hold. However, since HIV/AIDS has no known cure, preventative measure such as total abstinence, from heterosexual contact and use of condom should be encouraged. This will reduced the interaction function to zero and hence, the force of infection to zero, leading to a disease-free steady state. Other drug using therapies should also be encouraged to help extend the incubation period for AIDS so that infective will have more years of productive life before developing AIDS.

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