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A Discrete Mathematical Model for Homogeneous Population Density Dynamics of Single Weed Species

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Abstract: In this study, researchers employed the biological process to formulate a discrete-time homogeneous model for the dynamics of weed density interaction through biologically defined states and the mechanism of seedling recruitment incorporating weed reproduction from persistent seed bank within a crop growing season. Researchers obtained its steady-state solutions and analyzed them for local and global stabilities. Researchers discovered that the model is locally asymptotically stable but globally unstable. This result is contrary to the interesting property of the most standard biological one-dimensional discrete models which display global stability if they are locally stable. Although, the model equation falls within the category of population models that exhibit local stability but not globally stable. However, researchers conclude that the weed population may exhibit unexpected behaviours that is the population may not be predictable.

Key words: Biological process, discrete-time model, local stability, global stability, population density

INTRODUCTION

Weeds are generally defined, as uncultivated species that proliferate in agricultural setting thereby interfering with crop production. In fact, weed is a term applied to any plant that grows where it is not wanted, it is a human grouping of plants. They exist only in natural environments that have been disturbed by humans, such as agricultural lands, recreational, irrigation dams, etc. (Akobundu, 1987).

Population dynamics involve the study of population growth (numerical change in time) composition and spatial dispersion. The objectives are to identify the causes of numerical change in population and to explain how this cause act and interact to produce the observed pattern.

Not until recently population models (human and plants demography, infectious diseases, etc.) were concentrated mainly on the use of differential equations. Although, most populations, such as weeds and phylogenetically more evolved organisms live in seasonal environments and because of this have annual rhythms of reproduction and death. Besides, measurements are often made annually because interest is centred on population changes from year to year. Continuous differential equations are not well suited to these kinds of processes. Hence, there is need for other modelling techniques, especially when interested in population with only annual reproductive tendencies or predictable changes that occur seasonally. Discrete-time models are better suited for organism with annual or seasonal reproductive patterns (Allen *et al*., 1996; AlSharawi and Rhouma, 2010; Sacker, 2011). Since, plant has discrete generations (seasonal reproduction) difference dynamical equation systems are an appropriate mathematical tool to model behaviour of population with no overlapping generations, such as weeds. So, difference equations (discrete time model) are better suited for organisms and plants with seasonal reproductive pattern.

Although, the subject of difference equations and discrete models have been flourishing in the past 2 decades which have frequently been applied in models of annual plants (Allen *et al*., 1996) but density-dependent in discrete population models is partially understood.

It is well-established that population models can be derived from two different sources, data and biological process. The first rely completely upon data to look at the dynamics of the population (Rachel, 2011). The second is a model defined by biological processes which do not include any data but instead attempt to understand the dynamics of populations purely from what is expected to occur. These two types of models require different frame works, one needs a biology defined state and the other a data defined state.

In this study, researchers employed the biological process to develop discrete-time homogeneous models

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for the dynamics of weed density interaction through biologically defined states and the mechanism of seedling recruitment incorporating weed reproduction from persistent seed bank within a crop growing season.

MATERIALS AND METHODS

The development of weed population dynamics model has not been excessively prolific, so researchers employ the basic characteristic and life history of weed to formulate the model.

Characteristics and life cycle of weeds: A weed is usually characterized by rapid growth and it typically replaces other more desirable plants. Most weeds, particularly annuals, survive seasons of adverse weather and maintain their genetically heritable traits through seed production. The long-lived species tend to produce few seeds than the short-lived species that face more environmental hazards. The seed, therefore plays important roles in the survival and multiplication of weed, such as spread of the weed species (dispersal) weed seeds are dispersed in space and in time also protection during conditions that unfavourable for germination (dormancy) (Aldrich, 1984; Harper, 1977; Radosevich and Holt, 1984). Weed seeds are continuously added to agricultural land during cropping cycle. In the tropics, the bulk of the weed seeds are produced during the short period of bush fallow that fallow crop harvest. The quantity of seeds produce by given weed species in a growing season influences the richness of that species in a given habitat. It also, affects the competitiveness of the weed in crops grown in that location. Consequently, there has been increased weed pressure in agricultural lands in all parts of the tropics Africa where traditional long bush fallow periods have been replaced by short duration bush fallows.

Mature weed shed their seeds on agricultural lands and thus add to the population of weed seeds in or on the soil. This new and periodic weed seed supply has been described as the seed rain (Harper, 1977). Some of the weed seeds that are on or in the top soil may be killed by heat during pre-planting bush burning. Germinating weed seeds may, also be killed by re-emergence herbicides. Non-dormant weed seeds that escape these actions will germinate and the seedling weeds could be killed by adverse weather condition. Only those weed seedling that escape these action grow to maturity to produce mature seeds that add to the seed rain. It is this seed rain that replenished the reservoir of viable seeds in the soil has described this seed reservoir as the seed bank. It consists of seeds produced in a given area plus weed seeds that have migrated to the area as a result of the action of various agent of seed dissemination (Harper, 1977).

Formulation of the model: The following assumptions are taken into consideration in the formulation of the model equations:

- There are enough growth resources, e.g., nutrients, light, water, etc., that promote continuous growth of at least two plant species
- Within the populations of weed there are intra-specific competitions
- All parameters involved with the model formulation are non-negatives

Figure 1 depicts the life cycle of weeds. Researchers employ the assumptions, definition of variables and parameters as depicted in Fig. 1, as well as the procedure similar to what some other researchers have used to model plant population dynamics the difference equations which describe the dynamics of proliferation of single weed species is derived next (Allen *et al*., 1996; Nasir *et al*., 2012).

The residual seeds (y_t) in the year t consist of seeds from previous years (t-1) that have survived the dry season and remained dormant (y_{t-1}), as well as new seeds that have not germinated after surviving the dry season (x_{t-1}). Thus, y_t satisfys the equation:

$$y_t = P_d\left(1 - P_g\right)\left(x_{t-1} + y_{t-1}\right) \quad (1)$$

Maximum weed seed pool (S_t) in year t is:

$$S_t = x_t + y_t \quad (2)$$

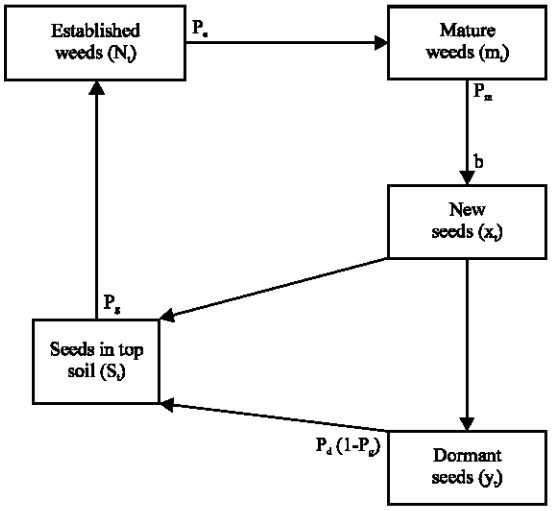

Fig. 1: A schematic model for population cycle of annual weeds

The established weed densities N_{t+1} in the year t+1 is made up of seed pool S_t in the top soil that survived the dry season, germinate and become established. This is describe by:

$$N_{t+1} = P_d P_g P_e S_t = P_d P_g P_e (x_t + y_t) \quad (3)$$

This is because the germination and establishment of new and residual seeds are two independent events. The new seeds n_t produced in year t are a function of matured weeds in the year t that is:

$$X_t = f(P_m, N_t) = P_m \frac{bN_t}{1 + aN_t} \quad (4)$$

Equation 4 is adaptation of Beaverton-Holts type function because researchers assumed a density dependent growth. Putting Eq. 1 and 4 into Eq. 3 gives:

$$N_{t+1} = P_d P_g P_e \left\{ P_m \frac{bN_t}{1 + aN_t} + P_d (1 - P_g)(x_t + y_t) \right\}$$
$$N_{t+1} = P_d P_g P_e P_m \frac{bN_t}{1 + aN_t} + P_d P_g P_e P_d (1 - P_g)(x_t + y_t) \quad (5)$$

Therefore, adopting Eq. 3 in Eq. 5 researchers have:

$$N_{t+1} = P_d P_g P_e P_m \frac{bN_t}{1 + aN_t} + P_d (1 - P_g) N_t \quad (6)$$

In a more compact form Eq. 6 becomes:

$$N_{t+1} = \frac{rbN_t}{1 + aN_t} + \gamma N$$

And subsequently written as:

$$N_{t+1} = \frac{rbN_t}{1 + aN_t} + \gamma N_t \quad (7)$$

Where:

r = $P_d P_g P_e P_m$
γ = $P_d (1-P_g)$
β = rb
b = Maximum seeds produced per mature weed
r = Intrinsic growth rate of weeds
β = Weed recruitment factor (i.e., fraction of seeds that germinate, become mature and produce seeds $\beta>0$
a = Crowding coefficient (equivalent to the intra-specific competitions)
γ = The density independent fraction of N_t surviving in the seed bank to the next season
$\beta/1+aN_t$ = The density-dependent net recruitment rate from generation to generation

This is a non-linear difference equation for the homogeneous population density of mature weeds in the year (t+1).

While the term $\beta N_t/1+aN_t$, a saturation function gives the population growth of the weed as a function of mature weed density. As the density increases the residual seeds in the top soil are denied access to enough growth resources (e.g., light, nutrient, water and space) thereby reduces the chances of seed germination and establishment. At low density, more seeds would have access to the growth resources and the proliferation of weeds follows. Therefore, Eq. 7 gives homogenous model for a single weed proliferation with no control.

RESULTS AND DISCUSSION

Analyses of the model equation: In this study, researchers analytically analyze the homogeneous weed proliferation Eq. 7 for the existence and stability of its associated steady-states (fixed-point) solutions.

The first step in understanding the dynamics of model population is to determine the steady-state (i.e., constant) solutions and the stability of the equilibrium (Cushing and Yicang, 1994). That is, usually the first step to take in order to study the dynamics of any system (model) is to find its steady-state points (solutions). So, the steady-state solutions of the models are obtained next.

Steady-state solutions of the model equation: A point is assumed to be a solution of the steady-state of the model equations only if all of its components are non-negative for biological and ecological significance. The methods used to solve for the steady-states in discrete model (difference equation) are comparable to those used in continuous models (differential equations) (Akinwande, 1999; Akinwande and Abdulrahman, 2011). To solve for the steady-state of Eq. 7, it is assumed that:

$$N_{t+1} = N_t, \text{implies that } \Delta N = N_{t+1} - N_t = 0$$
$$\text{So, let } N_{t+1} = N_t = \bar{N} \quad (8)$$

There are two non-negative solutions of the steady-states for the single species weed model Eq. 7. So, applying method Eq. 8, the steady-state of Eq. 7 satisfies the equation:

$$\bar{N} = \frac{\beta\bar{N}}{1+a\bar{N}} + \gamma\bar{N} \quad (9)$$

From Eq. 9, researchers have $\bar{N}$ (1-β/1+a$\bar{N}$-γ) = 0, one steady state solution is $\bar{N}=0$. That is non-zero steady-state occurs when F ($\bar{N}$) = 1. Hence, $\bar{N}$ = β-(1-γ)/a (1-γ). So, second fixed point (Eq. 7) exists and positive provided γ<1, since β>1-γ. Hence, the two non-negative steady-states are E_1 (0) and E_2 (β-(1-γ/a (1-γ).

The zero state E_1 is comparable to a situation of weed density extinct (or dies out) during dry season. While, the non-zero state E_2 is liken to the existence of weeds or infestation of weed in arable field.

Local stability of the steady-state solutions: To test for the local stability of the steady-states of the model equation, researchers adopted the well known stability theorem for discrete one-dimensional population models as stated by Cull (2007).

Theorem 1: If f (x) is differentiable, then a population model is locally stable if $|f'(\bar{x})|<1$ and if the model is locally stable then $|f'(\bar{x})|\leq 1$. Here, $\bar{x}$ is the unique equilibrium point of function x_{t+1} = f (x_t).

Applying this theorem to the model researchers obtain the derivative of RHS of Eq. 7, i.e., let:

$$f(\bar{N}) = \frac{\beta\bar{N}}{1+a\bar{N}} + \gamma\bar{N}$$

Then, after simplification researchers obtained:

$$f'(\bar{N}) = \frac{\beta}{(1+a\bar{N})^2} + \gamma \quad (10)$$

Stability of E_1 (0): Evaluating Eq. 10 at N = 0 gives f' (0) = β+γ. So, zero steady-states is locally stable if:

$$|f'(0)| = \beta + \gamma < 1 \quad (11)$$

Otherwise, it is not stable. Thus, if the model is stable, the density of the mature weed tends to zero and the weed population dies out or eradicated. Clearly, β+γ>1, since so E_1 = 0 is unstable. This implies that the density of the mature weed tends to a new equilibrium density E_2.

Stability of the solution E_2 ($\bar{N}$): Evaluating Eq. 11 at $\bar{N}$ = β+γ-1/a (1-γ) gives ((1-γ)^2/β)+γ. So, the steady-state E_2 is stable whenever ((1-γ)^2/β)+γ<1. That is:

$$\frac{(1-\gamma)}{\beta} < 1 \quad (12)$$

Proposition: If β+γ>1, then the non-zero steady-state E_2 is locally stable, otherwise it is not stable.

Proof: Suppose E_1 (0) is stable. It implies that β+γ<1. For E_2 to be stable, using Theorem 1, Eq. 12 must hold. That is:

$$\left|\frac{1-\gamma}{\beta}\right| < 1$$

Implies:

$$-1 < \left|\frac{1-\gamma}{\beta}\right| < 1$$

So:

$$-\beta < 1-\gamma < \beta$$

Then:

$$1-\beta < \gamma < 1+\beta$$

Implies:

$$1 < \beta+\gamma < 1+2\beta$$

This completes the proof. Hence, the non-zero steady-state E_2 is locally stable. While, E_1 (0) is unstable. Thus, the density of mature weeds N_t approaches or settles down to a positive constant value given by β+γ-1/a (1-γ).

Region of stability: The necessary condition for local stability implies that 1≤β+γ, 0≤β and 0≤γ≤1. But, if 0 =β and γ = 1 the model fitness function at the equilibrium $f(\bar{N})$ degenerates to $\bar{N}$ (i.e., $f(\bar{N})$ = $\bar{N}$) which is not a population model by the basic characteristics (the basic characteristics of the models reflect the growth of a population until it reaches some environmental carrying capacity and the subsequent decay of the population). So in the region of stability, the parameters satisfy the inequalities 1<β+γ, 0<β and 0≤γ<1.

Global stability of the non-trivial steady-state: It is important to know whether or not a model is globally stable. Models having this property are predictable while those that do not can exhibit unexpected behaviour (Heinschel, 1994). One of the tools used to prove global stability in difference equations is the schwarzian derivative which was first introduced into the study of one-dimensional dynamical system by David Singer (Heinschel, 1994; Liz, 2007).

$$S(f,x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)^2}{f'(x)}\right)$$

Calculation of the schwarzian for the model:

$$f'(\bar{N}) = \frac{\beta}{(1+a\bar{N})^2} + f''(\bar{N}) = \frac{-2a\beta}{(1+a\bar{N})^3}$$

and; $$f'''(\bar{N}) = \frac{6a^2\beta}{(1+a\bar{N})^4}$$

$$S(f,\bar{N}) = \frac{6a^2\beta}{(1+a\bar{N})^2\left[\beta+\gamma(1+a\bar{N})^2\right]} - \frac{3}{2}\left[\frac{-2a\beta}{(1+a\bar{N})\left[\beta+\gamma(1+a\bar{N})^2\right]}\right]^2$$

$$= \frac{6a^2\beta\left[\beta+\gamma(1+a\bar{N})^2\right] - 6a^2\beta^2}{\left\{(1+a\bar{N})\left[\beta+\gamma(1+a\bar{N})^2\right]\right\}^2}$$

$$= \frac{6a^2\beta\gamma(1+a\bar{N})^2}{\left\{(1+a\bar{N})\left[\beta+\gamma(1+a\bar{N})^2\right]\right\}^2}$$

This gives:

$$S(f,\bar{N}) = \frac{6a^2\beta\gamma}{\left[\beta+\gamma(1+a\bar{N})^2\right]^2} \quad (13)$$

This show that $s(f,\bar{N}) > 0$ everywhere. Hence, non-zero steady-state (E_2) is not globally stable. A good understanding of the behaviour of a model depends on knowing whether or not the model is globally stable. Hence, the weed population may exhibit unexpected behaviours (that is the population may not be predictable).

CONCLUSION

The steady-state solutions of the proposed model equation were obtained and analysed for local and global stabilities. The analysis shows that the model is locally asymptotically stable but globally unstable. This result is contrary to the interesting property of the most standard biological one-dimensional discrete models which display global stability if they are locally stable. Although, the model equation falls within the category of population models that exhibit local stability but not globally stable. Therefore, further study of the behaviour of this model for global stability is under consideration.

REFERENCES

Akinwande, N.I., 1999. A time-discrete mathematical model of yellow fever disease dynamics. J. Univ. Ghana, 12: 117-124.

Akinwande, N.I. and S. Abdulrahman, 2011. A time-discrete prey-predator population model. Proceedings of the 30th Annual Conference of the Nigerian Mathematical Society, (ACNMS'11), Minna, Nigeria.

Akobundu, I.O., 1987. Weed Science in the Tropics: Principles and Practices. John Wiley and Sons, New York.

AlSharawi, Z. and M.B.H. Rhouma, 2010. The discrete beverton-holt model with peri odic harvesting in a periodically fluctuating environment. Adv. Differ. Equat., 10.1155/2010/215875.

Aldrich, R.J., 1984. Weed Crop Ecology: Principle in Weed Management. Breton Publishers, Massachusetts, USA., Pages: 465.

Allen, L.J.S., E.J. Allen and S. Ponweera, 1996. A mathematical model for weed dispersal and control. Bull. Math. Biol., 58: 815-834.

Cull, P., 2007. Population models: Stability in one dimension. Bull. Math. Biol., 60: 989-1017.

Cushing, J.M. and Z. Yicang, 1994. The net reproductive value and stability in matrix population models. Nat. Resour. Modeling, 8: 297-333.

Harper, J.L., 1977. Population Biology of Plants. Academic Press, London, Pages: 892.

Heinschel, N., 1994. Sufficient conditions for global stability in population models. REU Summer Research Program, Oregon State University, USA.

Liz, E., 2007. Local stability implies global stability in some one discrete single-species models. Discr. Continuous Dyn. Syst. Ser. B, 7: 191-199.

Nasir, M.O., N.I. Akinwande and A. Sirajo, 2012. A mathematical model for the dynamics of single species of weed. Proceedings of the 31th Annual National Conference of Mathematical Association of Nigeria, October 2-5, 2012, Ahmadu Bello University Zaria, Kaduna State, Nigeria.

Rachel, B.H., 2011. Intergral projection models and analysis of patch dynamics of the reef building coral monstastraea annularis. Ph.D. Thesis, University of Exeter, UK.

Radosevich, S.R. and J.S. Holt, 1984. Weed Ecology: Implications for Vegetation Management. Wiley and Sons, New York, Pages: 265.

Sacker, R.J., 2011. Global stability in a multi-species periodic Leslie-Gower model. J. Biol. Dyn., 5: 549-562.